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Conformally parametrized surfaces associated with $\mathbb{C}P^{N-1}$ sigma models

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Abstract

Two-dimensional parametrized surfaces immersed in the $su(N)$ algebra are investigated. The focus is on surfaces parametrized by solutions of the equations for the $\mathbb{C}P^{N-1}$ sigma model. The Lie-point symmetries of the $\mathbb{C}P^{N-1}$ model are computed for arbitrary N. The Weierstrass formula for immersion is determined and an explicit formula for a moving frame on a surface is constructed. This allows us to determine the structural equations and geometrical properties of surfaces in $\mathbb{R}^{N^2-1}$. The fundamental forms, Gaussian and mean curvatures, Willmore functional and topological charge of surfaces are given explicitly in terms of any holomorphic solution of the $\mathbb{C}P^2$ model. The approach is illustrated through several examples, including surfaces immersed in low-dimensional $su(N)$ algebras.

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1. Introduction

Group theoretical methods have proven to be very useful for studying surfaces immersed in multi-dimensional spaces and for computing their main geometric characteristics [1–5]. It was shown in [6–9] that the problem of Weierstrass immersion of two-dimensional smooth surfaces in multi-dimensional Euclidean spaces is related to the surfaces in Lie algebras associated with the $\mathbb{C}P^{N-1}$ models. The main feature of this approach is that it allows one to replace the methods based on Dirac-type equations by a formalism connected with completely integrable $\mathbb{C}P^{N-1}$ models. The task of finding an increasing number of surfaces is related to choosing a suitable Lie representation of the $\mathbb{C}P^{N-1}$ model. Group analysis makes it possible to construct algorithms proceeding directly from the equations of the $\mathbb{C}P^{N-1}$ model and without referring to any additional considerations. The techniques for constructing two-dimensional surfaces immersed in $su(N)$ algebras, obtained from integrable models, are better understood for low-dimensional $\mathbb{C}P^{N-1}$ models. In that case, the geometric features of surfaces so obtained are interesting and the subject of ongoing study. A review of recent developments related to integrable models can be found in [10–13].

Over the last century and a half, the Weierstrass formula for the immersion of surfaces in Lie groups, Lie algebras and homogeneous spaces has been used extensively in various areas of mathematics, physics, chemistry and biology. We now list some of the most important examples.

In mathematics, the topic is of central importance in the formulation of the classical theory of surfaces. In particular, immersions are useful for studying surfaces with techniques of completely integrable continuous and discrete systems, as well as for the development and application of numerical tools [14, 15]. A description of the monodromy of solutions of Painlevé equations is yet another important application [16].

In physics, the concept has numerous applications in, e.g., two-dimensional gravity [17], field and string theory [18, 19], statistical physics (e.g., growth of crystals, surface waves, dynamics of vortex sheets, the two-body correlation function of the two-dimensional Ising model [20]), fluid dynamics (e.g., motion of boundaries between regions of differing densities and velocities [21]), plasma physics (geometry of magnetic surfaces and constant pressure surfaces in various fusion devices such as tokomaks, stellarators, magnetic mirrors [22]).

In chemistry, descriptions of energy and momentum transport along a polymer molecule constitute a significant area of application for the theory of immersions [23, 24]. In biology, the theory is frequently used in the study of the model for the Canham–Helfrich membrane and its continuous deformations [25, 26].

In general, the algebraic approach to the equations describing surface immersion has been proven to be very fruitful from a computational point of view. In addition, the geometric approach is of primary importance to the derivation and characterization of the governing equations for related phenomena in physics and other applied sciences.

This paper follows up on research in [6], where surfaces immersed in $su(N+1)$ algebras obtained via $\mathbb{C}P^N$ models were investigated. We generalize the results and also correct some formulae. To be precise, the new results presented in this paper include the Lie-point symmetry algebra of the $\mathbb{C}P^{N-1}$ model for arbitrary N. We also give new examples of surfaces immersed in a low-dimensional $su(N)$ algebra invariant under the scaling symmetries whose Gaussian curvature always vanishes. We delve deeply into the geometrical aspects of surfaces in $su(3)$ obtained from the $\mathbb{C}P^2$ model. For that case, we identify the moving frame and the structural equations, as well as the Willmore functional and the topological charge. The main goal of this paper is to provide a comprehensive, self-contained approach to the subject.

The paper is organized as follows. In section 2, we briefly review some basic notions and properties concerning the Euler–Lagrange equations associated with the $\mathbb{C}P^{N-1}$ models. In section 3, we discuss the Weierstrass formula for immersion in connection with the $\mathbb{C}P^{N-1}$ model, derive the induced metric and compute the scalar curvature. Section 4 is devoted to the Lie-point symmetries of the equations of the $\mathbb{C}P^{N-1}$ model for arbitrary N. Section 5 covers the analysis of the immersion of surfaces in the $su(3)$ algebra arising from the $\mathbb{C}P^2$ model. In section 6, we investigate the Weierstrass aspects for the immersion of surfaces in the $su(2)$ and $su(3)$ algebras which are associated with the $\mathbb{C}P^1$ and $\mathbb{C}P^2$ models, respectively. Section 7 deals with applications of the Weierstrass formula for the immersion of surfaces in the $su(2)$ and $su(3)$ algebras, as well as surfaces immersed in a low-dimensional $su(N)$ algebra invariant under the scaling symmetries.

2. The Euler–Lagrange equations associated with the $\mathbb{C}P^{N-1}$ sigma models

To keep the paper self-contained, we briefly review basic notions and properties of the $\mathbb{C}P^{N-1}$ sigma models (see, e.g., [10, 27, 28] and references therein). The domain of definition for the sigma model is assumed to be an open, connected and simply connected set $\Omega \subset \mathbb{C}$ with the Euclidean metric

$$ds^2 = d\xi\, d\bar{\xi} = (d\xi^1)^2 + (d\xi^2)^2, \qquad \xi = \xi^1 + i\xi^2, \tag{1}$$

where ξ and $\bar{\xi}$ are local coordinates in Ω. In the case of the $\mathbb{C}P^{N-1}$ models the target space is an $(N-1)$-dimensional complex projective space $\mathbb{C}P^{N-1}$, which is defined as the set of all complex lines in $\mathbb{C}^N$. The manifold structure on it is defined by an open covering

$$\mathcal{U}_k = \{[z] \mid z \in \mathbb{C}^N, z_k \neq 0\}, \qquad k = 1, \ldots, N, \tag{2}$$

where $[z] = \mathrm{span}\{z\}$ and the coordinate maps $h_k : \mathcal{U}_k \to \mathbb{C}^{N-1}$ are defined by

$$h_k(z) = \left(\frac{z_1}{z_k}, \ldots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \ldots, \frac{z_N}{z_k}\right). \tag{3}$$

We are interested in maps of the form $[z] : \Omega \to \mathbb{C}P^{N-1}$, which are stationary points of the action functional

$$S = \frac{1}{4}\int_\Omega (D_\mu z)^\dagger (D^\mu z)\, d\xi\, d\bar{\xi}, \qquad z^\dagger \cdot z = 1. \tag{4}$$

Here, D_μ and D^μ $(\mu = 1, 2)$ are the covariant derivatives acting on $z : \Omega \to \mathbb{C}^N$, defined by the formula

$$D_\mu z = \partial_\mu z - (z^\dagger \cdot \partial_\mu z) z, \tag{5}$$

where $\partial_\mu = \partial_{\xi^\mu}$. The action S does not depend on the choice of a representative of the class $[z]$. As usual, the symbol $\dagger$ denotes Hermitian conjugation, whereas the Hermitian inner product of $z = (z_1, \ldots, z_N)$ and $w = (w_1, \ldots, w_N)$ in $\mathbb{C}^N$ is denoted by

$$\langle z, w\rangle = z^\dagger \cdot w = \sum_{j=1}^N \bar{z}_j w_j. \tag{6}$$

Introducing

$$z = \frac{f}{|f|}, \qquad |f| = (f^\dagger \cdot f)^{\frac{1}{2}}, \tag{7}$$

the action functional (4) can be expressed as

$$S = \frac{1}{4}\int_\Omega \frac{1}{f^\dagger \cdot f}(\partial f^\dagger P \bar{\partial} f + \bar{\partial} f^\dagger P \partial f)\, d\xi\, d\bar{\xi}, \tag{8}$$

where ∂ and $\bar{\partial}$ denote the partial derivatives with respect to ξ and $\bar{\xi}$, respectively, i.e.,

$$\partial = \tfrac{1}{2}(\partial_{\xi^1} - i\partial_{\xi^2}), \qquad \bar{\partial} = \tfrac{1}{2}(\partial_{\xi^1} + i\partial_{\xi^2}). \tag{9}$$

The $N \times N$ matrix P is an orthogonal projector on the orthogonal complement of the complex line in $\mathbb{C}^N$. Therefore,

$$P = I_N - \frac{1}{f^\dagger \cdot f} f \otimes f^\dagger, \tag{10}$$

where I_N is the $N \times N$ identity matrix. Since P is an orthogonal projector it has the properties

$$P^\dagger = P, \qquad P^2 = P. \tag{11}$$

The map $[z]$ is determined by a solution of the Euler–Lagrange equations which are associated with the action (8). In the homogeneous coordinates f, the equations of motion take the form of a conservation law

$$\partial K - \bar{\partial} K^{\dagger} = 0, \tag{12}$$

where K and $K^{\dagger}$ are $N \times N$ matrices given by

$$\begin{aligned} K &= [\bar{\partial} P, P] = \frac{1}{f^{\dagger} \cdot f}(\bar{\partial} f \otimes f^{\dagger} - f \otimes \bar{\partial} f^{\dagger}) + \frac{f \otimes f^{\dagger}}{(f^{\dagger} \cdot f)^2}(\bar{\partial} f^{\dagger} \cdot f - f^{\dagger} \cdot \bar{\partial} f), \\ K^{\dagger} &= -[\partial P, P] = \frac{1}{f^{\dagger} \cdot f}(f \otimes \partial f^{\dagger} - \partial f \otimes f^{\dagger}) + \frac{f \otimes f^{\dagger}}{(f^{\dagger} \cdot f)^2}(\partial f^{\dagger} \cdot f - f^{\dagger} \cdot \partial f). \end{aligned} \tag{13}$$

Using the projector, the Euler–Lagrange equations (12) can also be written in the form of a conservation law

$$\partial[\bar{\partial} P, P] + \bar{\partial}[\partial P, P] = 0. \tag{14}$$

Through explicit calculation one can verify that the complex-valued functions

$$J = \frac{1}{f^{\dagger} \cdot f} \partial f^{\dagger} P \partial f, \qquad \bar{J} = \frac{1}{f^{\dagger} \cdot f} \bar{\partial} f^{\dagger} P \bar{\partial} f \tag{15}$$

satisfy

$$\bar{\partial} J = 0, \qquad \partial \bar{J} = 0, \tag{16}$$

for any solution f of the equations of motion (12).

Note that the action (4), as well as J and $\bar{J}$, are invariant under a global $U(N)$ transformation, i.e., $f \to uf$, where $u \in U(N)$. Due to this invariance, without loss of generality, we can set one of the components of the vector field f equal to 1. For instance, $f_1 = 1$. Consequently, the $\mathbb{C}P^{N-1}$ model can be expressed in one less variable through the relation

$$w_{i-1} = \frac{f_i}{f_1}, \qquad i = 2, \ldots, N-1. \tag{17}$$

3. The Weierstrass formula for immersion

For a given projector P satisfying the conservation law (14), we give the analytical description of a 2D smooth orientable surface $\mathcal{F}$ immersed in the $su(N)$ algebra. This is accomplished by constructing an exact $su(N)$ matrix-valued 1-form $\mathrm{d}X$ for which its 'potential', which is a matrix-valued 0-form X, determines a surface immersed in the $su(N)$ algebra. Once the 0-form X is calculated, we can treat the components of X as the coordinates of a surface in $su(N)$ and, hence, we can compute an explicit formula for immersion. In what follows, we shall refer to this as the generalized Weierstrass formula for immersion. Next, we investigate some geometrical properties of the surface $\mathcal{F}$ in the $su(N)$ algebra.

In order to construct and investigate surfaces in multi-dimensional spaces by analytical methods it is convenient to identify the $su(N)$ algebra with the (N^2-1)-dimensional Euclidean space through the relation

$$\mathbb{R}^{N^2-1} \simeq su(N). \tag{18}$$

For the sake of uniformity, we use the following definition of scalar product on $su(N)$

$$\langle A, B \rangle = -\tfrac{1}{2}\mathrm{tr}(AB), \tag{19}$$

where $A, B \in su(N)$.

Let us assume that the matrix K in (13) is constructed from a solution P of the Euler–Lagrange equation (14) defined on some connected and simply connected domain $\Omega \subset \mathbb{C}$. According to Poincaré's lemma, there then exists a closed matrix-valued 1-form,

$$dX = i(K^\dagger \, d\xi + K \, d\bar{\xi}), \tag{20}$$

which is also exact and takes its values in the $su(N)$ algebra of skew-Hermitian matrices. This means that X is a well-defined $su(N)$ real-valued function on Ω and

$$\partial X = iK^\dagger, \qquad \bar{\partial} X = iK. \tag{21}$$

It follows from the closedness of the 1-form dX that the integral

$$i \int_\gamma (K^\dagger \, d\xi + K \, d\bar{\xi}) = X(\xi, \bar{\xi}) \tag{22}$$

is locally independent of the path of integration. As a matter of fact, the integral only depends on the end points of the curve γ in $\mathbb{C}$.

The integral (22) defines a mapping

$$X : \Omega \ni (\xi, \bar{\xi}) \to X(\xi, \bar{\xi}) \in su(N), \tag{23}$$

which is called the generalized Weierstrass formula for immersion [6, 7].

As a consequence of (23), we can determine a surface $\mathcal{F}$ in $su(N)$ from a solution f of the Euler–Lagrange equation (12) defined on the domain $\Omega \subset \mathbb{C}$.

The complex tangent vectors to a surface $\mathcal{F}$ are given by (21) using (13). For the components of the induced metric one gets

$$\begin{aligned} &g_{\xi\xi} \equiv (\partial X, \partial X) = -J, \qquad g_{\bar{\xi}\bar{\xi}} \equiv (\bar{\partial} X, \bar{\partial} X) = -\bar{J}, \\ &g_{\xi\bar{\xi}} = g_{\bar{\xi}\xi} \equiv (\partial X, \bar{\partial} X) = q, \end{aligned} \tag{24}$$

where J and $\bar{J}$ are holomorphic functions defined in (15) and the quantity q is a positive real-valued function given by

$$q = \frac{1}{f^\dagger \cdot f} \bar{\partial} f^\dagger P \partial f \geqslant 0. \tag{25}$$

Thus, the first fundamental form of a surface $\mathcal{F}$ takes the form

$$I = -J \, d\xi^2 + 2q \, d\xi \, d\bar{\xi} - \bar{J} \, d\bar{\xi}^2. \tag{26}$$

Using the Schwartz inequality, it was shown in [6, 7] that this first fundamental form (26) is positive definite.

The scalar curvature is given by

$$\mathcal{K} = \frac{1}{2\sqrt{g}} \bar{\partial} \left[\frac{q}{\sqrt{g}} \partial \ln \left(-\frac{q^2}{J} \right) \right], \qquad \text{if} \quad J \neq 0 \tag{27}$$

and

$$\mathcal{K} = -q^{-1} \bar{\partial} \partial \ln q, \qquad \text{if} \quad J = 0, \tag{28}$$

where

$$g = \det(g_{ij}) = |J|^2 - q^2, \tag{29}$$

and the indices i and j stand for ξ and $\bar{\xi}$, respectively.

Let us now discuss the existence of certain classes of surfaces in the $su(N)$ algebra when the $\mathbb{C}P^{N-1}$ equations are subjected to specific differential constraints (DCs). These constraints allow us to reduce the overdetermined system to a system admitting first integrals. Doing so considerably simplifies the process of solving the initial $\mathbb{C}P^{N-1}$ equations (12).

Consequently, certain classes of non-splitting solutions can be constructed and they provide us with an explicit, simplified form of the Weierstrass formula for the immersion of a surface in $su(N)$.

Proposition 1. *If the complex-valued vector function*

$$\mathbb{C} \ni \xi \to f(\xi) \in \mathbb{C}^N \backslash \{0\} \tag{30}$$

satisfies both equations (12) for the $\mathbb{C}P^{N-1}$ *model equations and the differential constraints*

$$f^\dagger \cdot \partial f - \partial f^\dagger \cdot f = 0, \qquad f^\dagger \cdot \bar{\partial} f - \bar{\partial} f^\dagger \cdot f = 0, \tag{31}$$

then the generalized Weierstrass formula for the immersion of a surface $\mathcal{F}$ *in the* $su(N)$ *algebra has the form*

$$X(\xi, \bar{\xi}) = \mathrm{i} \int_\gamma \frac{f \otimes \partial f^\dagger - (\partial f^\dagger \cdot f)\widetilde{P}}{f^\dagger \cdot f} \, \mathrm{d}\xi + \frac{\bar{\partial} f \otimes f^\dagger - (f^\dagger \cdot \bar{\partial} f)\widetilde{P}}{f^\dagger \cdot f} \, \mathrm{d}\bar{\xi}, \tag{32}$$

where $\widetilde{P} = I_N - P$. *The first fundamental form is given by*

$$I = -J_1 \, \mathrm{d}\xi^2 + 2\left(\frac{\bar{\partial} f^\dagger \cdot \partial f}{f^\dagger \cdot f} - \frac{(\bar{\partial} f^\dagger \cdot f)(f^\dagger \cdot \partial f)}{(f^\dagger \cdot f)^2}\right) \mathrm{d}\xi \, \mathrm{d}\bar{\xi} - \bar{J}_1 \, \mathrm{d}\bar{\xi}^2, \tag{33}$$

where J_1 *and* $\bar{J}_1$ *are holomorphic functions,*

$$J_1 = \frac{\partial f^\dagger \cdot \partial f}{f^\dagger \cdot f} - \left(\frac{f^\dagger \cdot \partial f}{f^\dagger \cdot f}\right)^2, \qquad \bar{J}_1 = \frac{\bar{\partial} f^\dagger \cdot \bar{\partial} f}{f^\dagger \cdot f} - \left(\frac{\bar{\partial} f^\dagger \cdot f}{f^\dagger \cdot f}\right)^2, \tag{34}$$

which satisfy

$$\bar{\partial} J_1 = 0, \qquad \partial \bar{J}_1 = 0, \tag{35}$$

whenever (12) and (31) hold.

Proof. If we append the two DCs in (31) to the $\mathbb{C}P^{N-1}$ equations (12) then the matrices K and $K^\dagger$ in (13) become

$$\begin{aligned} K_1 &= \frac{1}{f^\dagger \cdot f}(\bar{\partial} f \otimes f^\dagger - f \otimes \bar{\partial} f^\dagger), \\ K_1^\dagger &= \frac{1}{f^\dagger \cdot f}(f \otimes \partial f^\dagger - \partial f \otimes f^\dagger). \end{aligned} \tag{36}$$

Hence, the Weierstrass formula for immersion takes the form

$$\begin{aligned} X(\xi, \bar{\xi}) &= \mathrm{i} \int_\gamma \left(K_1^\dagger \mathrm{d}\xi + K_1 \mathrm{d}\bar{\xi}\right) \\ &= \mathrm{i} \int_\gamma \frac{f \otimes \partial f^\dagger - \partial f \otimes f^\dagger}{f^\dagger \cdot f} \, \mathrm{d}\xi + \frac{\bar{\partial} f \otimes f^\dagger - f \otimes \bar{\partial} f^\dagger}{f^\dagger \cdot f} \, \mathrm{d}\bar{\xi}. \end{aligned} \tag{37}$$

On the other hand, from (12), we are able to deduce that the matrix K can be decomposed as

$$K = M + L, \tag{38}$$

where

$$M = (I_N - P)\bar{\partial} P, \qquad L = -\bar{\partial} P(I_N - P). \tag{39}$$

It can be shown that the matrices M and L satisfy the same conservation laws (12) as the matrix K, e.g.,

$$\partial M = \bar{\partial} M^\dagger, \qquad \partial L = \bar{\partial} L^\dagger. \tag{40}$$

Note that the two conservation laws in (40) are not independent since M and L differ by a total divergence,

$$M = L + \bar{\partial} P. \tag{41}$$

Taking into account the overdetermined system composed of the conservation laws (12) and DCs (31) for the function f, the matrices M and L become

$$\begin{aligned} M_1 &= -\frac{f \otimes \bar{\partial} f^\dagger - (f^\dagger \cdot \bar{\partial} f)\widetilde{P}}{f^\dagger \cdot f}, \qquad & M_1^\dagger &= -\frac{\partial f \otimes f^\dagger - (\partial f^\dagger \cdot f)\widetilde{P}}{f^\dagger \cdot f}, \\ L_1 &= \frac{\bar{\partial} f \otimes f^\dagger - (f^\dagger \cdot \bar{\partial} f)\widetilde{P}}{f^\dagger \cdot f}, \qquad & L_1^\dagger &= \frac{f \otimes \partial f^\dagger - (\partial f^\dagger \cdot f)\widetilde{P}}{f^\dagger \cdot f}. \end{aligned} \tag{42}$$

As a consequence of the conservation laws (40) for the matrices M_1 and L_1, the Weierstrass formula for immersion (22) takes the following simple form:

$$\begin{aligned} X(\xi, \bar{\xi}) &= \mathrm{i} \int_\gamma \left(M_1^\dagger \mathrm{d}\xi + M_1 \mathrm{d}\bar{\xi} \right) \\ &= -\mathrm{i} \int_\gamma \frac{\partial f \otimes f^\dagger - (\partial f^\dagger \cdot f)\widetilde{P}}{f^\dagger \cdot f} \mathrm{d}\xi + \frac{f \otimes \bar{\partial} f^\dagger - (f^\dagger \cdot \bar{\partial} f)\widetilde{P}}{f^\dagger \cdot f} \mathrm{d}\bar{\xi}, \end{aligned} \tag{43}$$

or

$$\begin{aligned} X(\xi, \bar{\xi}) &= \mathrm{i} \int_\gamma \left(L_1^\dagger \mathrm{d}\xi + L_1 \mathrm{d}\bar{\xi} \right) \\ &= \mathrm{i} \int_\gamma \frac{f \otimes \partial f^\dagger - (\partial f^\dagger \cdot f)\widetilde{P}}{f^\dagger \cdot f} \mathrm{d}\xi + \frac{\bar{\partial} f \otimes f^\dagger - (f^\dagger \cdot \bar{\partial} f)\widetilde{P}}{f^\dagger \cdot f} \mathrm{d}\bar{\xi}, \end{aligned} \tag{44}$$

respectively. As a consequence of (41), (43) and (44), it can be shown that the two different Weierstrass data $\left(L_1, L_1^\dagger\right)$ or $\left(M_1, M_1^\dagger\right)$ correspond to different parametrizations of the same surface $\mathcal{F}$ in the $su(N)$ algebra.

In this case, the quantity J takes the simple form

$$J_1 = \frac{\partial f^\dagger \cdot \partial f}{f^\dagger \cdot f} - \left(\frac{f^\dagger \cdot \partial f}{f^\dagger \cdot f} \right)^2, \qquad \bar{J}_1 = \frac{\bar{\partial} f^\dagger \cdot \bar{\partial} f}{f^\dagger \cdot f} - \left(\frac{\bar{\partial} f^\dagger \cdot f}{f^\dagger \cdot f} \right)^2. \tag{45}$$

Using the conservation laws (12) and DCs (31) for the function f, we find that J_1 is a holomorphic function, e.g., $\bar{\partial} J_1 = 0$ whenever (12) and (31) hold. As a consequence of (43), (44) and (45), the components of the induced metric are

$$g_{\xi\xi} = -J_1, \qquad g_{\bar{\xi}\bar{\xi}} = -\bar{J}_1, \qquad g_{\xi\bar{\xi}} = \frac{\bar{\partial} f^\dagger \cdot \partial f}{f^\dagger \cdot f} - \frac{(\bar{\partial} f^\dagger \cdot f)(f^\dagger \cdot \partial f)}{(f^\dagger \cdot f)^2}, \tag{46}$$

which completes the proof. □

Remark. If the $\mathbb{C}P^1$ model is subjected to the two DCs in (31), then the determinant of the induced metric g vanishes, hence, we obtain a curve instead of a surface. However, the determinant of the induced metric g of the $\mathbb{C}P^{N-1}$ model does not vanish for $N \geqslant 3$.

Note that the complex-valued vector function $\mathbb{C} \ni \xi \to f(\xi) \in \mathbb{C}^N \backslash \{0\}$ is a holomorphic $(\bar{\partial} f = 0)$ solution of the $\mathbb{C}P^{N-1}$ model (12) if and only if the generalized Weierstrass formula for the immersion of a surface $\mathcal{F}$ has the skew-Hermitian form

$$X(\xi, \bar{\xi}) = -\mathrm{i}P \in su(N). \tag{47}$$

If f is holomorphic, i.e., $\bar{\partial} f = 0$, then by virtue of equations (39) and the differential consequences of the identity $(I_N - P)P = 0$, we obtain

$$M = 0, \qquad \bar{\partial} P P = 0. \tag{48}$$

Using the differential consequences for the projector P, we get

$$\begin{aligned} &\bar{\partial} P P = 0, \qquad && P \partial P = 0, \\ &\bar{\partial} P = P \bar{\partial} P, \qquad && \partial P = \partial P P. \end{aligned} \tag{49}$$

Substituting (49) into (13), we obtain

$$K = -\bar{\partial} P, \qquad K^{\dagger} = -\partial P. \tag{50}$$

Hence, the Weierstrass formula for immersion (22) of $\mathcal{F}$ is expressed in terms of the projector P and is a skew-Hermitian matrix given by (47). This result coincides with that obtained in [29].

The converse is also true. Indeed, if we assume that the Weierstrass formula for the immersion (22) of $\mathcal{F}$ is a projector P then the differential of X leads to (50). Using the differential consequences of the relation $P^2 = P$, we obtain the relations (49) which lead to $M = 0$. In view of equations (39), this implies that, in the generic case, solutions of the $\mathbb{C}P^{N-1}$ model (12) must be holomorphic.

Also, note that in the case of the holomorphic solutions of the $\mathbb{C}P^{N-1}$ model the corresponding complex-valued function (15) vanishes, i.e.,

$$J = \frac{1}{f^{\dagger} \cdot f} \partial f^{\dagger} P \partial f = 0. \tag{51}$$

An analogous statement can be made for anti-holomorphic solutions ($\partial f = 0$) of equation (12). For this case, we have

$$L = 0, \qquad P \bar{\partial} P = 0, \qquad \partial P P = 0. \tag{52}$$

Hence, from (13), the matrices K and $K^{\dagger}$ become

$$K = \bar{\partial} P, \qquad K^{\dagger} = \partial P. \tag{53}$$

Finally, one can see that the Weierstrass formula for the immersion of $\mathcal{F}$ is the skew-Hermitian form

$$X(\xi, \bar{\xi}) = \mathrm{i} P \in su(N). \tag{54}$$

4. The Lie-point symmetries of the $\mathbb{C}P^{N-1}$ sigma models

In this section, we present the explicit formulae for the Lie-point symmetries of the $\mathbb{C}P^{N-1}$ model (12) for arbitrary N. To do so, we first compute the symmetries for the $\mathbb{C}P^1, \mathbb{C}P^2$ and $\mathbb{C}P^3$ models. We then generalize the results to the $\mathbb{C}P^{N-1}$ case by induction. For the computation of the Lie-point symmetries, we search for the most general (point) transformations of the independent and dependent variables which leave the solution set of (12) invariant. Locally, such transformations are given by a vector field of the form [30]

$$\vec{v} = \eta^1 \partial + \eta^2 \bar{\partial} + \sum_{j=1}^{N-1} \Phi_j^1 \partial_{w_j} + \sum_{j=1}^{N-1} \Phi_j^2 \partial_{\bar{w}_j}, \tag{55}$$

where η^1, η^2, Φ_j^1 and Φ_j^2 are functions of $\xi, \bar{\xi}$ and the affine coordinates $w_1, \bar{w}_1, \ldots, w_{N-1}, \bar{w}_{N-1}$. According to the symmetry criterion [30], the second prolongation of $\vec{v}$ acting on (12) must vanish on the solution set of (12). This requirement leads to the so-called determining equations, whose solution yields the functions η^1, η^2, Φ_j^1 and Φ_j^2.

Generating the determining equations is entirely algorithmic. Reducing and solving them can be done by fully automatic with sophisticated software or, perhaps more reliably, by interactively adding information extracted from the simplest determining equations before computing the full set. Many software packages have been written to perform Lie symmetry computations. In-depth reviews of such packages can be found in [31–34].

For low dimensions, e.g., for $N \leqslant 4$, we did the computations independently with SYMMGRP.MAX and by hand. For the latter, we took advantage of the discrete symmetries of the model. For the $\mathbb{C}P^{N-1}$ models with $N \geqslant 4$, after eliminating all single-term determining equations and their differential consequences, we were left with several hundred of determining equations. Using SYMMGRP.MAX interactively, these determining equations were further reduced and eventually completely solved.

We now discuss the Lie-point symmetries of the $\mathbb{C}P^{N-1}$ models for $N = 2, 3$ and 4, separately.

The equations for the $\mathbb{C}P^1$ model, expressed in terms of the homogeneous coordinate w_1 defined in (17), are given by

$$\partial\bar{\partial}w_1 - \frac{2\bar{w}_1}{A_1}\partial w_1\bar{\partial}w_1 = 0, \qquad \partial\bar{\partial}\bar{w}_1 - \frac{2w_1}{A_1}\partial\bar{w}_1\bar{\partial}\bar{w}_1 = 0, \tag{56}$$

where $A_1 = 1 + w_1\bar{w}_1$. The general solution of the determining equations associated with vector field (55) is given by

$$\begin{aligned} &\eta^1 = \eta^1(\xi), \qquad \eta^2 = \eta^2(\bar{\xi}),\\ &\Phi_1^1 = \alpha_1 w_1^2 + \beta_1 w_1 + \gamma_1,\\ &\Phi_1^2 = \gamma_1\bar{w}_1^2 - \beta_1\bar{w}_1 + \alpha_1, \end{aligned} \tag{57}$$

where η^1 and η^2 are arbitrary functions of ξ and $\bar{\xi}$, respectively, and α_1, β_1 and γ_1 are arbitrary constants. Thus, the corresponding symmetry algebra $\mathcal{L}_1$ is spanned by five generators, namely,

$$\begin{aligned} &X_1 = \eta^1(\xi)\partial, \qquad X_2 = \eta^2(\bar{\xi})\bar{\partial},\\ &X_3 = w_1^2\partial_{w_1} + \partial_{\bar{w}_1},\\ &X_4 = w_1\partial_{w_1} - \bar{w}_1\partial_{\bar{w}_1},\\ &X_5 = \partial_{w_1} + \bar{w}_1^2\partial_{\bar{w}_1}. \end{aligned} \tag{58}$$

The algebra $\mathcal{L}_1$ can be decomposed as a direct sum of two infinite-dimensional simple Lie algebras and the $su(2)$ algebra generated by $\{X_3, X_4, X_5\}$, i.e.,

$$\mathcal{L}_1 = \{X_1\} \oplus \{X_2\} \oplus su(2). \tag{59}$$

Likewise, in terms of homogeneous coordinates w_1 and w_2 in (17), the equations for the $\mathbb{C}P^2$ model read

$$\begin{aligned} &\partial\bar{\partial}w_1 - \frac{2\bar{w}_1}{A_2}\partial w_1\bar{\partial}w_1 - \frac{\bar{w}_2}{A_2}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) = 0,\\ &\partial\bar{\partial}w_2 - \frac{2\bar{w}_2}{A_2}\partial w_2\bar{\partial}w_2 - \frac{\bar{w}_1}{A_2}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) = 0,\\ &\partial\bar{\partial}\bar{w}_1 - \frac{2w_1}{A_2}\partial\bar{w}_1\bar{\partial}\bar{w}_1 - \frac{w_2}{A_2}(\bar{\partial}\bar{w}_1\partial\bar{w}_2 + \partial\bar{w}_1\bar{\partial}\bar{w}_2) = 0,\\ &\partial\bar{\partial}\bar{w}_2 - \frac{2w_2}{A_2}\partial\bar{w}_2\bar{\partial}\bar{w}_2 - \frac{w_1}{A_2}(\bar{\partial}\bar{w}_1\partial\bar{w}_2 + \partial\bar{w}_1\bar{\partial}\bar{w}_2) = 0, \end{aligned} \tag{60}$$

where $A_2 = 1 + w_1\bar{w}_1 + w_2\bar{w}_2$. Upon integration, the determining equations yield

$$\begin{aligned}
&\eta^1 = \eta^1(\xi), \qquad \eta^2 = \eta^2(\bar{\xi}),\\
&\Phi_1^1 = k_1 w_1^2 + k_2 w_1 w_2 + k_4 w_1 + k_5 w_2 + k_6,\\
&\Phi_2^1 = k_2 w_2^2 + k_1 w_1 w_2 + k_3 w_2 + k_7 w_1 + k_8,\\
&\Phi_1^2 = k_6 \bar{w}_1^2 + k_8 \bar{w}_1 \bar{w}_2 - k_4 \bar{w}_1 - k_7 \bar{w}_2 + k_1,\\
&\Phi_2^2 = k_8 \bar{w}_2^2 + k_6 \bar{w}_1 \bar{w}_2 - k_3 \bar{w}_2 - k_5 \bar{w}_1 + k_2,
\end{aligned} \tag{61}$$

where k_i $(i = 1, \ldots, 8)$ are arbitrary constants. The associated symmetry algebra $\mathcal{L}_2$ of (60) is thus spanned by the following ten generators:

$$\begin{aligned}
&X_1 = \eta^1(\xi)\partial, \qquad X_2 = \eta^2(\bar{\xi})\bar{\partial},\\
&X_3 = w_1^2\partial_{w_1} + w_1 w_2 \partial_{w_2} + \partial_{\bar{w}_1},\\
&X_4 = w_1 w_2 \partial_{w_1} + w_2^2 \partial_{w_2} + \partial_{\bar{w}_2},\\
&X_5 = w_2\partial_{w_2} - \bar{w}_2\partial_{\bar{w}_2},\\
&X_6 = w_1\partial_{w_1} - \bar{w}_1\partial_{\bar{w}_1},\\
&X_7 = w_2\partial_{w_1} - \bar{w}_1\partial_{\bar{w}_2},\\
&X_8 = \partial_{w_1} + \bar{w}_1^2\partial_{\bar{w}_1} + \bar{w}_1\bar{w}_2\partial_{\bar{w}_2},\\
&X_9 = w_1\partial_{w_2} - \bar{w}_2\partial_{\bar{w}_1},\\
&X_{10} = \partial_{w_2} + \bar{w}_1\bar{w}_2\partial_{\bar{w}_1} + \bar{w}_2^2\partial_{\bar{w}_2}.
\end{aligned} \tag{62}$$

As in the previous case, the symmetry algebra $\mathcal{L}_2$ can be decomposed as a direct sum of two infinite-dimensional simple Lie algebras and the $su(3)$ algebra.

In like fashion, in terms of w_1, w_2 and w_3 in (17), the equations for the $\mathbb{C}P^3$ model are

$$\begin{aligned}
&\partial\bar{\partial}w_1 - \frac{2\bar{w}_1}{A_3}\partial w_1\bar{\partial}w_1 - \frac{\bar{w}_2}{A_3}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) - \frac{\bar{w}_3}{A_3}(\partial w_1\bar{\partial}w_3 + \bar{\partial}w_1\partial w_3) = 0,\\
&\partial\bar{\partial}w_2 - \frac{2\bar{w}_2}{A_3}\partial w_2\bar{\partial}w_2 - \frac{\bar{w}_1}{A_3}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) - \frac{\bar{w}_3}{A_3}(\partial w_2\bar{\partial}w_3 + \bar{\partial}w_2\partial w_3) = 0,\\
&\partial\bar{\partial}w_3 - \frac{2\bar{w}_3}{A_3}\partial w_3\bar{\partial}w_3 - \frac{\bar{w}_1}{A_3}(\partial w_1\bar{\partial}w_3 + \bar{\partial}w_1\partial w_3) - \frac{\bar{w}_2}{A_3}(\partial w_2\bar{\partial}w_3 + \bar{\partial}w_2\partial w_3) = 0,\\
&\partial\bar{\partial}\bar{w}_1 - \frac{2w_1}{A_3}\partial \bar{w}_1\bar{\partial}\bar{w}_1 - \frac{w_2}{A_3}(\partial \bar{w}_1\bar{\partial}\bar{w}_2 + \bar{\partial}\bar{w}_1\partial \bar{w}_2) - \frac{w_3}{A_3}(\partial \bar{w}_1\bar{\partial}\bar{w}_3 + \bar{\partial}\bar{w}_1\partial \bar{w}_3) = 0,\\
&\partial\bar{\partial}\bar{w}_2 - \frac{2w_2}{A_3}\partial \bar{w}_2\bar{\partial}\bar{w}_2 - \frac{w_1}{A_3}(\partial \bar{w}_1\bar{\partial}\bar{w}_2 + \bar{\partial}\bar{w}_1\partial \bar{w}_2) - \frac{w_3}{A_3}(\partial \bar{w}_2\bar{\partial}\bar{w}_3 + \bar{\partial}\bar{w}_2\partial \bar{w}_3) = 0,\\
&\partial\bar{\partial}\bar{w}_3 - \frac{2w_3}{A_3}\partial \bar{w}_3\bar{\partial}\bar{w}_3 - \frac{w_1}{A_3}(\partial \bar{w}_1\bar{\partial}\bar{w}_3 + \bar{\partial}\bar{w}_1\partial \bar{w}_3) - \frac{w_2}{A_3}(\partial \bar{w}_2\bar{\partial}\bar{w}_3 + \bar{\partial}\bar{w}_2\partial \bar{w}_3) = 0,
\end{aligned} \tag{63}$$

where $A_3 = 1 + w_1\bar{w}_1 + w_2\bar{w}_2 + w_3\bar{w}_3$. After straightforward but long calculations the determining equations yield

$$\begin{aligned}
&\eta^1 = \eta^1(\xi), \qquad \eta^2 = \eta^2(\bar{\xi}),\\
&\Phi_1^1 = c_1 w_1^2 + c_2 w_1 w_2 + c_3 w_1 w_3 + c_7 w_1 + c_{10} w_2 + c_{11} w_3 + c_4,\\
&\Phi_2^1 = c_2 w_2^2 + c_1 w_1 w_2 + c_3 w_2 w_3 + c_{13} w_1 + c_8 w_2 + c_{12} w_3 + c_5,\\
&\Phi_3^1 = c_3 w_3^2 + c_1 w_1 w_3 + c_2 w_2 w_3 + c_{14} w_1 + c_{15} w_2 + c_9 w_3 + c_6,\\
&\Phi_1^2 = c_4 \bar{w}_1^2 + c_5 \bar{w}_1 \bar{w}_2 + c_6 \bar{w}_1 \bar{w}_3 - c_7 \bar{w}_1 - c_{13} \bar{w}_2 - c_{14} \bar{w}_3 + c_1,
\end{aligned}$$

$$\Phi_2^2 = c_5\bar{w}_2^2 + c_4\bar{w}_1\bar{w}_2 + c_6\bar{w}_2\bar{w}_3 - c_{10}\bar{w}_1 - c_8\bar{w}_2 - c_{15}\bar{w}_3 + c_2,$$
$$\Phi_3^2 = c_6\bar{w}_3^2 + c_4\bar{w}_1\bar{w}_3 + c_5\bar{w}_2\bar{w}_3 - c_{11}\bar{w}_1 - c_{12}\bar{w}_2 - c_9\bar{w}_3 + c_3, \tag{64}$$

where c_i $(i = 1, \ldots, 15)$ are arbitrary constants. Hence, the generators corresponding to the symmetry algebra $\mathcal{L}_3$ of (63) are given by

$$\begin{aligned}
&X_1 = \eta^1(\xi)\partial, \qquad X_2 = \eta^2(\bar{\xi})\bar{\partial},\\
&S_i = w_i\partial_{w_i} - \bar{w}_i\partial_{\bar{w}_i},\\
&T_{ij} = w_i\partial_{w_j} - \bar{w}_j\partial_{\bar{w}_i}, \qquad i \neq j,\\
&Y_i = w_i^2\partial_{w_i} + \sum_{j\neq i}^{3} w_i w_j\partial_{w_j} + \partial_{\bar{w}_i},\\
&Z_i = \bar{w}_i^2\partial_{\bar{w}_i} + \sum_{j\neq i}^{3} \bar{w}_i\bar{w}_j\partial_{\bar{w}_j} + \partial_{w_i},
\end{aligned} \tag{65}$$

where $i, j = 1, 2, 3$. From S_i, Y_i and Z_i we get nine generators; from T_{ij} we obtain six generators. The symmetry algebra $\mathcal{L}_3$ can be written as a direct sum of two infinite-dimensional simple Lie algebras and $su(4)$. The results for the low-dimensional cases reveal an emerging pattern: the symmetry algebra is a direct sum of two infinite-dimensional Lie algebras and a finite-dimensional one. Furthermore, the finite-dimensional part of the symmetry algebras for the $\mathbb{C}P^1, \mathbb{C}P^2$ and $\mathbb{C}P^3$ models is associated with the $su(2), su(3)$ and $su(4)$ algebras, respectively.

We now turn to the $\mathbb{C}P^{N-1}$ model for arbitrary N. In homogeneous coordinates w_i, the equations are

$$\begin{aligned}
&\partial\bar{\partial}w_i - \frac{2\bar{w}_i}{A_{N-1}}\partial w_i\bar{\partial}w_i - \frac{1}{A_{N-1}}\sum_{j\neq i}^{N-1}\bar{w}_j(\partial w_i\bar{\partial}w_j + \bar{\partial}w_i\partial w_j) = 0,\\
&\partial\bar{\partial}\bar{w}_i - \frac{2w_i}{A_{N-1}}\partial\bar{w}_i\bar{\partial}\bar{w}_i - \frac{1}{A_{N-1}}\sum_{j\neq i}^{N-1}w_j(\partial\bar{w}_i\bar{\partial}\bar{w}_j + \bar{\partial}\bar{w}_i\partial\bar{w}_j) = 0,
\end{aligned} \tag{66}$$

where $i = 1, 2, \ldots, N-1$ and $A_{N-1} = 1 + \sum_i^{N-1} w_i\bar{w}_i$.

By induction, it can be shown that the symmetry algebra $\mathcal{L}_{N-1}$ of (66) is generated by

$$\begin{aligned}
&X_1 = \eta^1(\xi)\partial, \qquad X_2 = \eta^2(\bar{\xi})\bar{\partial},\\
&S_i = w_i\partial_{w_i} - \bar{w}_i\partial_{\bar{w}_i},\\
&T_{ij} = w_i\partial_{w_j} - \bar{w}_j\partial_{\bar{w}_i}, \qquad i \neq j,\\
&Y_i = w_i^2\partial_{w_i} + \sum_{j\neq i}^{N-1} w_i w_j\partial_{w_j} + \partial_{\bar{w}_i},\\
&Z_i = \bar{w}_i^2\partial_{\bar{w}_i} + \sum_{j\neq i}^{N-1} \bar{w}_i\bar{w}_j\partial_{\bar{w}_j} + \partial_{w_i},
\end{aligned} \tag{67}$$

where $i, j = 1, 2, \ldots, N-1$. Furthermore, it can be shown that the symmetry algebra $\mathcal{L}_{N-1}$ is a direct sum of two infinite-dimensional Lie algebras and the $su(N)$ algebra, i.e.,

$$\mathcal{L}_{N-1} = \{X_1\} \oplus \{X_2\} \oplus su(N). \tag{68}$$

Finally, we consider two limiting cases:

(1) If $w_{N-1} \to 0$ then the $\mathbb{C}P^{N-1}$ model reduces to the $\mathbb{C}P^{N-2}$ model. Also, if all $N-2$ homogeneous coordinates vanish, then the $\mathbb{C}P^{N-1}$ model reduces to the $\mathbb{C}P^1$ model.

(2) If $w_i \to \frac{w}{\sqrt{N-1}}$ for $i = 1, \ldots, N-1$, then the $\mathbb{C}P^{N-1}$ model reduces to the $\mathbb{C}P^1$ model.

Hence, in the $\mathbb{C}P^1$ case, we have a significant simplification.

5. Immersion of surfaces into the $su(3)$ algebra arising from the $\mathbb{C}P^2$ sigma model

In this section, we explore the metric aspects of surfaces immersed in the $su(3)$ algebra associated with the holomorphic (anti-holomorphic) solutions of the $\mathbb{C}P^2$ model. From the properties of the Hermitian matrix ∂K we determine explicitly a moving frame on a conformally parametrized surface $\mathcal{F}$ in $\mathbb{R}^8$. We also derive the corresponding Gauss–Weingarten equations expressed in terms of any holomorphic solution of the $\mathbb{C}P^2$ model. This investigation is a follow-up to earlier work [6, 7]. It allows us to communicate our new insights into the subject, as well as to present additional geometric characteristics of surfaces obtained from the model.

The assumption that the set $\{w_1, w_2\}$ is a holomorphic solution of the equations for the $\mathbb{C}P^2$ model implies that the quantity J in (15) vanishes. The induced metric on $\mathcal{F}$ given in (26) is then conformal. In the $\mathbb{C}P^2$ case, the 3×3 projector matrix in (10) reads

$$P = I_3 - \frac{1}{A_2}\begin{pmatrix} 1 & w_1 & w_2 \\ \bar{w}_1 & w_1\bar{w}_1 & w_2\bar{w}_1 \\ \bar{w}_2 & w_1\bar{w}_2 & w_2\bar{w}_2 \end{pmatrix}, \tag{69}$$

where I_3 is the 3×3 identity matrix. Assume that we are dealing with the generic case. That is, where the projector P is a solution of the Euler–Lagrange equations (60) such that the induced metric g has a non-vanishing determinant in some neighborhood of a regular point $(\xi_0, \bar{\xi}_0) \in \Omega \subset \mathbb{C}$. Further assume that a conformally parametrized surface $\mathcal{F}$, given by (22) and associated with the $\mathbb{C}P^2$ model, is described by a moving frame on $\mathcal{F}$ in $\mathbb{R}^8$:

$$\vec{\tau} = (\eta_1 = \partial X, \eta_2 = \bar{\partial} X, \eta_3, \ldots, \eta_8)^T, \tag{70}$$

where superscript T stands for transpose. Here, the vectors $\eta_1, \ldots, \eta_8$ are identified with 3×3 skew-Hermitian matrices through the isomorphism (18). Furthermore, assume that the vectors form an orthonormal set,

$$(\eta_j, \eta_k) = \delta_{jk}, \qquad j, k = 1, \ldots, 8, \tag{71}$$

where δ_{jk} is the Kronecker delta. Due to the normalization of the $su(3)$-valued function X on Ω, we can express the moving frame in (70) on $\mathcal{F}$ in terms of the adjoint $SU(3)$ representation. In the neighborhood of a regular point $p = (\xi_0, \bar{\xi}_0) \in \mathbb{C}$ an orthonormal moving frame $\vec{\tau}$ on $\mathcal{F}$ satisfies

$$\begin{aligned} \eta_1 &= \mathrm{i}\mathrm{e}^{\frac{u}{2}}\phi^\dagger y_-\phi, \qquad & \eta_2 &= \mathrm{i}\mathrm{e}^{\frac{u}{2}}\phi^\dagger y_+\phi, \\ \eta_j &= \phi^\dagger s_j\phi, & j &= 3, \ldots, 8, \end{aligned} \tag{72}$$

where u is a real-valued function of ξ and $\bar{\xi}$. The function ϕ in (72) belongs to $SU(3)$ and can be decomposed into the product of three $SU(2)$ factors, i.e.,

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & b_1 \\ 0 & -\bar{b}_1 & \bar{a}_1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi}\cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \mathrm{e}^{-\mathrm{i}\varphi}\cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2 & b_2 \\ 0 & -\bar{b}_2 & \bar{a}_2 \end{pmatrix}, \tag{73}$$

where $a_i, b_i, i = 1, 2$, are complex-valued functions of ξ and $\bar{\xi}$, subject to the constraints $|a_i|^2 + |b_i|^2 = 1$ and α, φ are real-valued functions of $\xi, \bar{\xi} \in \mathbb{C}$. Here, the set $\{s_1, \ldots, s_8\}$

forms an orthonormal basis of the Lie algebra $su(3)$ (e.g., the so-called Gell–Mann matrices [35]) given by

$$s_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad s_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix},$$
$$s_4 = \frac{1}{\sqrt{3}} \begin{pmatrix} -2\mathrm{i} & 0 & 0 \\ 0 & \mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}, \qquad s_5 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad s_6 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{74}$$
$$s_7 = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad s_8 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}.$$

These matrices satisfy the following trace condition:

$$\mathrm{tr}(s_i s_j) = -2\delta_{ij}. \tag{75}$$

We also introduced the following notation:

$$y_- = \frac{\mathrm{i}}{2}(s_1 - \mathrm{i}s_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad y_+ = \frac{\mathrm{i}}{2}(s_1 + \mathrm{i}s_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{76}$$

As a direct consequence of the moving frame (72) we get

$$(\phi^\dagger y_- \phi)^\dagger = \phi^\dagger y_+ \phi. \tag{77}$$

Note that, over the space $\mathbb{R}$, the set $\{y_-, y_+\}$ spans the same space as $\{s_1, s_2\}$.

Requiring that the parametrization of a surface $\mathcal{F}$ be conformal leads to the following conditions:

$$\begin{aligned} g_{\xi\xi} &= (\partial X, \partial X) = -\tfrac{1}{2}\mathrm{e}^u \mathrm{tr}(y_-)^2 = 0, \\ g_{\bar\xi\bar\xi} &= (\bar\partial X, \bar\partial X) = -\tfrac{1}{2}\mathrm{e}^u \mathrm{tr}(y_+)^2 = 0, \\ g_{\xi\bar\xi} &= (\partial X, \bar\partial X) = \tfrac{1}{2}\mathrm{e}^u \mathrm{tr}(y_- y_+) = \tfrac{1}{2}\mathrm{e}^u, \end{aligned} \tag{78}$$

and

$$\begin{aligned} (\partial X, \eta_j) &= -\tfrac{1}{2}\mathrm{e}^{\frac{u}{2}} \mathrm{tr}(y_- s_j) = 0, \\ (\bar\partial X, \eta_j) &= -\tfrac{1}{2}\mathrm{e}^{\frac{u}{2}} \mathrm{tr}(y_+ s_j) = 0, \\ (\eta_j, \eta_k) &= -\tfrac{1}{2}\mathrm{tr}(s_j s_k) = \delta_{jk}, \end{aligned} \tag{79}$$

where $j, k = 3, \ldots, 8$. Thus, we have the following proposition.

Proposition 2. *In the adjoint $SU(3)$ representation, the moving frame (72) of a conformally parametrized surface $\mathcal{F}$ is described in terms of holomorphic solutions $\{w_1, w_2\}$ of the $\mathbb{C}P^2$ equations (60) by the formulae*

$$\eta_1 = -\frac{\mathrm{i}}{{A_2}^2} \begin{pmatrix} \delta & \beta & \gamma \\ \bar w_1 \delta & \bar w_1 \beta & \bar w_1 \gamma \\ \bar w_2 \delta & \bar w_2 \beta & \bar w_2 \gamma \end{pmatrix}, \qquad \eta_2 = -\frac{\mathrm{i}}{{A_2}^2} \begin{pmatrix} \bar\delta & w_1 \bar\delta & w_2 \bar\delta \\ \bar\beta & w_1 \bar\beta & w_2 \bar\beta \\ \bar\gamma & w_1 \bar\gamma & w_2 \bar\gamma \end{pmatrix}, \tag{80}$$

and

$$u = \ln\left(\frac{\rho}{{A_2}^2}\right), \tag{81}$$

where we define

$$\begin{aligned}
\delta &= \bar{w}_1\partial w_1 + \bar{w}_2\partial w_2,\\
\beta &= w_1\bar{w}_2\partial w_2 - (1+|w_2|^2)\partial w_1,\\
\gamma &= \bar{w}_1 w_2\partial w_1 - (1+|w_1|^2)\partial w_2,\\
\rho &= |\partial w_1|^2 + |\partial w_2|^2 + |w_2\partial w_1 - w_1\partial w_2|^2.
\end{aligned} \tag{82}$$

Proof. Using the polar decomposition of the $SU(3)$ group given by (73), and calculating the products in the frame (72), yields

$$\begin{aligned}
\eta_1 &= \mathrm{i}\mathrm{e}^{\frac{u}{2}}\begin{pmatrix} -a_1b_1\sin^2\alpha & -b_1\sin\alpha\zeta & -b_1\sin\alpha\mu\\ \chi a_1\sin\alpha & \chi\zeta & \chi\mu\\ \nu a_1\sin\alpha & \nu\zeta & \nu\mu\end{pmatrix},\\
\eta_2 &= \mathrm{i}\mathrm{e}^{\frac{u}{2}}\begin{pmatrix} -\bar{a}_1\bar{b}_1\sin^2\alpha & \bar{\chi}\bar{a}_1\sin\alpha & \bar{\nu}\bar{a}_1\sin\alpha\\ -\bar{b}_1\sin\alpha\bar{\zeta} & \bar{\chi}\bar{\zeta} & \bar{\nu}\bar{\zeta}\\ -\bar{b}_1\sin\alpha\bar{\mu} & \bar{\chi}\bar{\mu} & \bar{\nu}\bar{\mu}\end{pmatrix},
\end{aligned} \tag{83}$$

where

$$\begin{aligned}
\chi &= -a_1b_2 - \bar{a}_2b_1\,\mathrm{e}^{\mathrm{i}\varphi}\cos\alpha, & \zeta &= -b_1\bar{b}_2 + a_1a_2\,\mathrm{e}^{-\mathrm{i}\varphi}\cos\alpha,\\
\mu &= \bar{a}_2b_1 + a_1b_2\,\mathrm{e}^{-\mathrm{i}\varphi}\cos\alpha, & \nu &= a_1a_2 - b_1\bar{b}_2\,\mathrm{e}^{\mathrm{i}\varphi}\cos\alpha.
\end{aligned} \tag{84}$$

Comparing (80) with (83) we obtain an underdetermined system of eight equations for nine unknown functions $a_i, b_i \in \mathbb{C}$, $i = 1, 2$, and $\alpha, \varphi, u \in \mathbb{R}$. This system has a unique solution up to a $U(1)$ transformation. In other words, the phase $\mathrm{e}^{\mathrm{i}\varphi}$ remains arbitrary.

A straightforward algebraic computation gives a_i, b_i and α in terms of the fields w_1 and w_2 for the $\mathbb{C}P^2$ model. Explicitly,

$$\begin{aligned}
a_1 &= \frac{\sqrt{\delta\kappa}}{A_2\sin\alpha}\mathrm{e}^{-u/4}, & b_1 &= \frac{\sqrt{\delta/\kappa}}{A_2\sin\alpha}\mathrm{e}^{-u/4},\\
a_2 &= -\frac{\mathrm{e}^{\mathrm{i}\varphi}\bar{\partial}\bar{w}_2(w_2\partial w_1 - w_1\partial w_2)}{\rho\sin\alpha\cos\alpha}, & b_2 &= \frac{\mathrm{e}^{\mathrm{i}\varphi}\bar{\partial}\bar{w}_1(w_2\partial w_1 - w_1\partial w_2)}{\rho\sin\alpha\cos\alpha},\\
\sin^2\alpha &= \frac{|\partial w_1|^2 + |\partial w_2|^2}{\rho}, & \cos^2\alpha &= \frac{|w_2\partial w_1 - w_1\partial w_2|^2}{\rho},
\end{aligned} \tag{85}$$

with u as in (81) and

$$\kappa = \frac{\delta\cos\alpha}{w_2\partial w_1 - w_1\partial w_2}\mathrm{e}^{-\mathrm{i}\varphi}. \tag{86}$$

With the above, we can determine the moving frame (72) on $\mathcal{F}$, expressed in terms of w_1 and w_2, in the required form (80). That ends the proof since by direct computation one can check that the compatibility conditions, i.e., $\partial\bar{\partial}X = \bar{\partial}\partial X$, for (72) are trivially satisfied. □

Remark. The explicit expressions for the complex normals $\eta_3, \ldots, \eta_8$ to this surface immersed in $su(3)$ have been calculated. However, the resulting expressions (in terms of w_1 and w_2) are rather involved. A specific example is given in appendix.

The real-valued function u given by (85) represents the total energy [27] of the $\mathbb{C}P^2$ model defined over S^2, since

$$u = 2\ln(|Dz|^2 + |\bar{D}z|^2) \tag{87}$$

holds.

Using the components of the induced metric (26), we can write the nonzero Christoffel symbols of the second kind as

$$\Gamma^1_{11} = \frac{1}{q}\partial q, \qquad \Gamma^2_{22} = \frac{1}{q}\bar{\partial} q. \tag{88}$$

In this case, q defined in (25) becomes

$$q = \frac{|\partial w_1|^2 + |\partial w_2|^2 + |w_1\partial w_2 - w_2\partial w_1|^2}{2(1+|w_1|^2+|w_2|^2)^2}. \tag{89}$$

Finally, taking into account (71), (78) and (79), the moving frame (70) on $\mathcal{F}$ satisfies the following Gauss–Weingarten equations:

$$\begin{aligned}\partial^2 X &= \frac{\partial q}{q}\partial X + J_j\eta_j,\\ \partial\bar{\partial} X &= H_j\eta_j,\\ \partial\eta_j &= -2\frac{A_2^2}{\rho}(H_j\partial X + J_j\bar{\partial}X) + S_{jk}\eta_k,\end{aligned} \tag{90}$$

and

$$\begin{aligned}\bar{\partial}^2 X &= \frac{\bar{\partial} q}{q}\bar{\partial} X + \bar{J}_j\eta_j,\\ \bar{\partial}\partial X &= H_j\eta_j,\\ \bar{\partial}\eta_j &= -2\frac{A_2^2}{\rho}(\bar{J}_j\partial X + H_j\bar{\partial}X) + \bar{S}_{jk}\eta_k,\end{aligned} \tag{91}$$

where

$$J_j = -\tfrac{1}{2}\mathrm{tr}(\partial^2 X\eta_j), \qquad H_j = -\tfrac{1}{2}\mathrm{tr}(\partial\bar{\partial}X\eta_j), \tag{92}$$

and

$$S_{jk} + S_{kj} = 0, \qquad \bar{S}_{jk} + \bar{S}_{kj} = 0, \qquad j \neq k = 3, \ldots, 8. \tag{93}$$

The Gauss–Codazzi–Ricci equations, which are the compatibility conditions for (90) and (91), coincide with the equations of the $\mathbb{C}P^{N-1}$ model. However, the explicit forms of the coefficients H_j and J_j depend locally on the chosen orthonormal basis $\{\eta_3, \ldots, \eta_8\}$ of the space normal to the surface $\mathcal{F}$ at a given point $p = (\xi_0, \bar{\xi}_0) \in X$. Note that quantities H_j and J_j are not completely arbitrary. Using (78) and the fact that $J = 0$, it becomes clear that the complex tangent vectors have to satisfy the following differential constraints:

$$(\partial^2 X, \bar{\partial}\partial X) = 0, \qquad (\bar{\partial}^2 X, \partial\bar{\partial}X) = 0. \tag{94}$$

For any holomorphic solution $(w_i, \bar{w}_i), i = 1, 2,$ of the $\mathbb{C}P^2$ model, we computed explicitly the form of the first and second fundamental forms, I and II, and the mean curvature vector $\mathcal{H}$ of a conformally parametrized surface $\mathcal{F}$ at some regular point $p = (\xi_0, \bar{\xi}_0) \in X$. They are

$$\begin{aligned}\mathrm{I} &= \frac{\rho}{A_2^2}\mathrm{d}\xi\,\mathrm{d}\bar{\xi},\\ \mathrm{II} &= \left(\partial^2 X - \frac{\partial q}{q}\partial X\right)\mathrm{d}\xi^2 + 2\partial\bar{\partial}X\mathrm{d}\xi\,\mathrm{d}\bar{\xi} + \left(\bar{\partial}^2 X - \frac{\bar{\partial} q}{q}\bar{\partial}X\right)\mathrm{d}\bar{\xi}^2,\\ \mathcal{H} &= \frac{2}{q}\partial\bar{\partial}X,\end{aligned} \tag{95}$$

respectively. The second derivatives of the Weierstrass representation X can be computed from (83).

One can also compute some of the global properties of surfaces associated with the $\mathbb{C}P^2$ sigma model, using the well-known formulae (see, e.g., [36, 37]). For instance, for any set of holomorphic solutions $(w_i, \bar{w}_i)$, $i = 1, 2$, of the $\mathbb{C}P^2$ model, the Willmore functional assumes the form

$$W = -4\mathrm{i}\int_\Omega \frac{1}{q}[\partial P, \bar{\partial} P]^2\, \mathrm{d}\xi\, \mathrm{d}\bar{\xi}, \tag{96}$$

whose values depend only on the fields and their derivatives on the boundary $\partial\Omega$ of the open set Ω.

Under the above assumptions and provided that the $\mathbb{C}P^2$ model is defined on the whole Riemannian sphere S^2, the topological charge becomes

$$Q = -\frac{1}{8\pi}\int_{S^2} q\, \mathrm{d}\xi\, \mathrm{d}\bar{\xi}. \tag{97}$$

If the above integral exists, then it is an integer which globally characterizes the surface.

6. The Weierstrass formula for the immersion of surfaces in the $su(2)$ and $su(3)$ algebras

In this section we apply the general idea of the Weierstrass representation of surfaces given in section 3 to two specific cases, namely, the $\mathbb{C}P^1$ and $\mathbb{C}P^2$ models. For each case, we first find the concrete form of the generalized Weierstrass representation of surfaces associated with these models and then we give the corresponding Weierstrass data for the holomorphic solutions.

It is known [6, 7] that, with the projector P given by (10), one can compute explicitly the formula for immersion (22) in terms of the complex fields w_i of the equations of motion of the model.

We start with the case $N = 2$. The orthogonal projector P and matrix K are then given by

$$P = I_2 - \frac{1}{A_1}\begin{pmatrix} 1 & w_1 \\ \bar{w}_1 & w_1\bar{w}_1 \end{pmatrix}, \tag{98}$$

and

$$K = \frac{1}{A_1^2}\begin{pmatrix} \bar{w}_1\bar{\partial}w_1 - w_1\bar{\partial}\bar{w}_1 & -\left(\bar{\partial}w_1 + w_1^2\bar{\partial}\bar{w}_1\right) \\ (\bar{\partial}\bar{w}_1 + \bar{w}_1^2\bar{\partial}w_1) & w_1\bar{\partial}\bar{w}_1 - \bar{w}_1\bar{\partial}w_1 \end{pmatrix}, \tag{99}$$

where, as usual, w_1 is the homogeneous coordinate defined by (17). Based on the expression of the matrix K for the $\mathbb{C}P^1$ model, the Weierstrass data follow from (20). In order to obtain real-valued 1-forms we decompose $\mathrm{d}X$ given in (20) into its real and imaginary parts,

$$\mathrm{d}X = \mathrm{d}X^1 + \mathrm{i}\mathrm{d}X^2. \tag{100}$$

So,

$$\begin{aligned} \mathrm{d}X^1 &= \frac{\mathrm{i}}{2}[(K^\dagger - \bar{K})\,\mathrm{d}\xi + (K - K^T)\,\mathrm{d}\bar{\xi}], \\ \mathrm{d}X^2 &= \frac{1}{2}[(K^\dagger + \bar{K})\,\mathrm{d}\xi + (K + K^T)\,\mathrm{d}\bar{\xi}]. \end{aligned} \tag{101}$$

It is easily seen that $\mathrm{d}X^1$ is skew-symmetric and $\mathrm{d}X^2$ is symmetric. Realizing that the 2D surface associated with the $\mathbb{C}P^1$ model is immersed in the $su(2)$ algebra, the two real-valued

1-forms can also be expressed in terms of the Pauli matrices. Since $\mathrm{d}X^1$ is skew-symmetric and $\mathrm{d}X^2$ is symmetric, the 1-forms can be represented as

$$\mathrm{d}X^1 = i\,\mathrm{d}X_2\sigma_2, \qquad \mathrm{d}X^2 = \mathrm{d}X_1\sigma_1 + \mathrm{d}X_3\sigma_3, \tag{102}$$

where σ_1, σ_2 and σ_3 are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{103}$$

After substituting the matrix K from (99) into (101) and comparing the result with (102), it is easy to see that the real-valued 1-forms $\mathrm{d}X_i$, $i = 1, 2, 3$, can be expressed in terms of the solutions of the Euler–Lagrange equations of the $\mathbb{C}P^1$ model. Indeed,

$$\begin{aligned}
\mathrm{d}X_1 &= \frac{1}{2A_1^2}\left(\left[\left(1-\bar{w}_1^2\right)\partial w_1 - \left(1-w_1^2\right)\partial\bar{w}_1\right]\mathrm{d}\xi + \text{c.c.}\right),\\
\mathrm{d}X_2 &= \frac{\mathrm{i}}{2A_1^2}\left(\left[\left(1+w_1^2\right)\partial\bar{w}_1 + \left(1+\bar{w}_1^2\right)\partial w_1\right]\mathrm{d}\xi - \text{c.c.}\right),\\
\mathrm{d}X_3 &= \frac{1}{A_1^2}([w_1\partial\bar{w}_1 - \bar{w}_1\partial w_1]\,\mathrm{d}\xi + \text{c.c.}),
\end{aligned} \tag{104}$$

where 'c.c.' denotes the complex conjugate. In fact, these real-valued 1-forms constitute the generalized Weierstrass formula for immersion for the $\mathbb{C}P^1$ model.

Now, we further restrict ourselves to the holomorphic solutions of the $\mathbb{C}P^1$ model. This restriction is necessary if the model is defined on S^2 with a finite action [27]. Using holomorphic solutions, $\mathrm{d}X_i$, $i = 1, 2, 3$, can be reduced into

$$\begin{aligned}
\mathrm{d}X_1 &= \frac{1}{2}\partial\left(\frac{w_1+\bar{w}_1}{A_1}\right)\mathrm{d}\xi + \text{c.c.},\\
\mathrm{d}X_2 &= \frac{\mathrm{i}}{2}\left[\partial\left(\frac{w_1-\bar{w}_1}{A_1}\right)\mathrm{d}\xi - \text{c.c.}\right],\\
\mathrm{d}X_3 &= -\partial\left(\frac{|w_1|^2}{A_1}\right)\mathrm{d}\xi + \text{c.c.}
\end{aligned} \tag{105}$$

Integration gives

$$X_1 = \frac{w_1+\bar{w}_1}{2A_1}, \qquad X_2 = \mathrm{i}\frac{w_1-\bar{w}_1}{2A_1}, \qquad X_3 = -\frac{|w_1|^2}{A_1}, \tag{106}$$

where the constants of integration are set to zero.

It is well known that the 2D surface associated with the holomorphic solutions of the $\mathbb{C}P^1$ model is the surface of a sphere [27]. Confirmation of that result follows from (106). Indeed, upon elimination of w_1 and $\bar{w}_1$, we obtain

$$X_1^2 + X_2^2 + \left(X_3 + \frac{1}{2}\right)^2 = \frac{1}{4}. \tag{107}$$

So, all points of the 2D surface lie on the surface of a sphere of radius 1/2 centered at $(0, 0, -1/2)$.

We now consider the case $N = 3$. The corresponding orthogonal projector P is given in (69) and matrix $K = -\mathrm{i}\eta_2$ with η_2 in (80). Since the 2D surface associated with the $\mathbb{C}P^2$ model is immersed in the $su(3)$ algebra, the two real-valued 1-forms, $\mathrm{d}X^1$ and $\mathrm{d}X^2$, obtained by decomposing $\mathrm{d}X = \mathrm{i}(K^\dagger\,\mathrm{d}\xi + K\,\mathrm{d}\bar{\xi})$ into real and imaginary parts, can be expressed in terms of the orthonormal basis of the Lie algebra $su(3)$. Keeping in mind that $\mathrm{d}X^1$ is skew-symmetric and $\mathrm{d}X^2$ is symmetric, the real-valued 1-forms are given by

$$
\begin{aligned}
\mathrm{d}X^1 &= \mathrm{d}X_2 s_2 + \mathrm{d}X_5 s_5 + \mathrm{d}X_6 s_6,\\
\mathrm{d}X^2 &= \mathrm{i}(\mathrm{d}X_1 s_1 + \mathrm{d}X_3 s_3 + \mathrm{d}X_4 s_4 + \mathrm{d}X_7 s_7 + \mathrm{d}X_8 s_8),
\end{aligned} \tag{108}
$$

where the Gell–Mann matrices s_i, $i = 1, \ldots, 8$, are given in (74).

Using $K = -\mathrm{i}\eta_2$ and comparing (101) with (108), it follows that the real-valued 1-forms $\mathrm{d}X_i$, $i = 1, \ldots, 8$, can be expressed in terms of the solutions of the Euler–Lagrange equations of the $\mathbb{C}P^2$ model as

$$
\begin{aligned}
\mathrm{d}X_1 &= \frac{1}{2A_2^2}\Big(\big[\big(w_2^2 - w_1^2\big)(\bar{w}_1\partial\bar{w}_2 - \bar{w}_2\partial\bar{w}_1) - \big(\bar{w}_2^2 - \bar{w}_1^2\big)(w_1\partial w_2 - w_2\partial w_1)\\
&\quad - w_2\partial\bar{w}_1 + \bar{w}_2\partial w_1 - w_1\partial\bar{w}_2 + \bar{w}_1\partial w_2\big]\mathrm{d}\xi + \mathrm{c.c.}\Big),\\
\mathrm{d}X_2 &= \frac{\mathrm{i}}{2A_2^2}\Big(\big[\big(w_1^2 + w_2^2\big)(\bar{w}_2\partial\bar{w}_1 - \bar{w}_1\partial\bar{w}_2) + \big(\bar{w}_1^2 + \bar{w}_2^2\big)(w_2\partial w_1 - w_1\partial w_2)\\
&\quad + w_2\partial\bar{w}_1 + \bar{w}_2\partial w_1 - w_1\partial\bar{w}_2 - \bar{w}_1\partial w_2\big]\mathrm{d}\xi - \mathrm{c.c.}\Big),\\
\mathrm{d}X_3 &= \frac{1}{2A_2^2}\Big(\big[w_2\partial\bar{w}_2 - w_1\partial\bar{w}_1 - \bar{w}_2\partial w_2 + \bar{w}_1\partial w_1\\
&\quad + 2|w_1|^2(w_2\partial\bar{w}_2 - \bar{w}_2\partial w_2) - 2|w_2|^2(w_1\partial\bar{w}_1 - \bar{w}_1\partial w_1)\big]\mathrm{d}\xi + \mathrm{c.c.}\Big),\\
\mathrm{d}X_4 &= \frac{\sqrt{3}}{2A_2^2}\Big(\big[w_1\partial\bar{w}_1 + w_2\partial\bar{w}_2 - \bar{w}_1\partial w_1 - \bar{w}_2\partial w_2\big]\mathrm{d}\xi + \mathrm{c.c.}\Big),\\
\mathrm{d}X_5 &= -\frac{\mathrm{i}}{2A_2^2}\Big(\big[\big(1 + \bar{w}_1^2 + |w_2|^2\big)\partial w_1 + \big(1 + w_1^2 + |w_2|^2\big)\partial\bar{w}_1\\
&\quad + (w_2\partial\bar{w}_2 - \bar{w}_2\partial w_2)(w_1 - \bar{w}_1)\big]\mathrm{d}\xi - \mathrm{c.c.}\Big),\\
\mathrm{d}X_6 &= -\frac{\mathrm{i}}{2A_2^2}\Big(\big[\big(1 + \bar{w}_2^2 + |w_1|^2\big)\partial w_2 + \big(1 + w_2^2 + |w_1|^2\big)\partial\bar{w}_2\\
&\quad + (w_1\partial\bar{w}_1 - \bar{w}_1\partial w_1)(w_2 - \bar{w}_2)\big]\mathrm{d}\xi - \mathrm{c.c.}\Big),\\
\mathrm{d}X_7 &= \frac{1}{2A_2^2}\Big(\big[\big(1 - w_1^2 + |w_2|^2\big)\partial\bar{w}_1 - \big(1 - \bar{w}_1^2 + |w_2|^2\big)\partial w_1\\
&\quad + (\bar{w}_2\partial w_2 - w_2\partial\bar{w}_2)(w_1 + \bar{w}_1)\big]\mathrm{d}\xi + \mathrm{c.c.}\Big),\\
\mathrm{d}X_8 &= \frac{1}{2A_2^2}\Big(\big[\big(1 - w_2^2 + |w_1|^2\big)\partial\bar{w}_2 - \big(1 - \bar{w}_2^2 + |w_1|^2\big)\partial w_2\\
&\quad + (\bar{w}_1\partial w_1 - w_1\partial\bar{w}_1)(w_2 + \bar{w}_2)\big]\mathrm{d}\xi + \mathrm{c.c.}\Big).
\end{aligned} \tag{109}
$$

These eight real-valued 1-forms constitute the generalized Weierstrass formula for immersion for the $\mathbb{C}P^2$ model.

Remark. Note that the reflection transformations in independent or dependent variables and their complex conjugates preserve the form of the $\mathbb{C}P^2$ model. So does the generalized $SU(2)$ transformation. Indeed, if the complex-valued functions u_1 and u_2 are solutions of the $\mathbb{C}P^2$ model, then the complex-valued functions w_1 and w_2 defined by the generalized $SU(2)$ transformation,

$$\begin{aligned} w_1 &\to \frac{a^2u_1 - b^2u_2 - \sqrt{2}ab}{\sqrt{2}(a\bar{b}u_1 + \bar{a}bu_2) + |a|^2 - |b|^2}, \\ w_2 &\to \frac{-\bar{b}^2u_1 + \bar{a}^2u_2 - \sqrt{2}\bar{a}\bar{b}}{\sqrt{2}(a\bar{b}u_1 + \bar{a}bu_2) + |a|^2 - |b|^2}, \end{aligned} \tag{110}$$

for $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$, are also solutions of the $\mathbb{C}P^2$ model.

These transformations can be used to restrict the range of parameters appearing in the explicit form of solutions of the $\mathbb{C}P^2$ model. They allow one to simplify the Weierstrass representation.

Again, we restrict ourselves to the holomorphic solutions of the $\mathbb{C}P^2$ model. In that case, the eight real-valued 1-forms $\mathrm{d}X_i$, $i = 1, \ldots, 8$, are

$$\begin{aligned} \mathrm{d}X_1 &= \frac{1}{2}\partial\left(\frac{w_1\bar{w}_2 + \bar{w}_1w_2}{A_2}\right)\mathrm{d}\xi + \text{c.c.}, \\ \mathrm{d}X_2 &= \frac{\mathrm{i}}{2}\left[\partial\left(\frac{w_1\bar{w}_2 - \bar{w}_1w_2}{A_2}\right)\mathrm{d}\xi - \text{c.c.}\right], \\ \mathrm{d}X_3 &= \frac{1}{2}\partial\left(\frac{|w_1|^2 - |w_2|^2}{A_2}\right)\mathrm{d}\xi + \text{c.c.}, \\ \mathrm{d}X_4 &= -\frac{\sqrt{3}}{2}\partial\left(\frac{|w_1|^2 + |w_2|^2}{A_2}\right)\mathrm{d}\xi + \text{c.c.}, \\ \mathrm{d}X_5 &= -\frac{\mathrm{i}}{2}\left[\partial\left(\frac{w_1 - \bar{w}_1}{A_2}\right)\mathrm{d}\xi - \text{c.c.}\right], \\ \mathrm{d}X_6 &= -\frac{\mathrm{i}}{2}\left[\partial\left(\frac{w_2 - \bar{w}_2}{A_2}\right)\mathrm{d}\xi - \text{c.c.}\right], \\ \mathrm{d}X_7 &= -\frac{1}{2}\partial\left(\frac{w_1 + \bar{w}_1}{A_2}\right)\mathrm{d}\xi + \text{c.c.}, \\ \mathrm{d}X_8 &= -\frac{1}{2}\partial\left(\frac{w_2 + \bar{w}_2}{A_2}\right)\mathrm{d}\xi + \text{c.c.} \end{aligned} \tag{111}$$

Ignoring integration constants, after integration we obtain

$$\begin{aligned} &X_1 = \frac{w_1\bar{w}_2 + \bar{w}_1w_2}{2A_2}, && X_2 = \mathrm{i}\frac{w_1\bar{w}_2 - \bar{w}_1w_2}{2A_2}, && X_3 = \frac{|w_1|^2 - |w_2|^2}{2A_2}, \\ &X_4 = -\sqrt{3}\frac{|w_1|^2 + |w_2|^2}{2A_2}, && X_5 = -\mathrm{i}\frac{w_1 - \bar{w}_1}{2A_2}, && X_6 = -\mathrm{i}\frac{w_2 - \bar{w}_2}{2A_2}, \\ &X_7 = -\frac{w_1 + \bar{w}_1}{2A_2}, && X_8 = -\frac{w_2 + \bar{w}_2}{2A_2}, \end{aligned} \tag{112}$$

which determines the coordinates of the radius vector $\vec{X} = (X_1, \ldots, X_8)$ of a two-dimensional surface in $\mathbb{R}^8$.

Note that in the limiting cases $w_i \to w/\sqrt{2}, i = 1, 2$, or $w_1 \to 0$ or $w_2 \to 0$, the generalized Weierstrass formula (109) for the immersion of the $\mathbb{C}P^2$ model reduces (after straightforward manipulations) to the generalized Weierstrass formula (104) for the immersion of the $\mathbb{C}P^1$ model. Consequently, the coordinates of radius vector $\vec{X}$ in (112) for the holomorphic solutions of the $\mathbb{C}P^2$ model then reduce to the coordinates of $\vec{X}$ in (106) for the holomorphic solutions of the $\mathbb{C}P^1$ model.

When dealing with the 2D surface associated with the holomorphic solutions of the $\mathbb{C}P^2$ model, all points lie on the affine sphere,

$$4X_1^2 + 4X_2^2 + 4X_3^2 + \tfrac{2}{\sqrt{3}}X_4 + X_5^2 + X_6^2 + X_7^2 + X_8^2 = 0. \tag{113}$$

It is straightforward to show that the coordinates of the radius vector (112) satisfy (113).

7. Examples of surfaces associated with the $\mathbb{C}P^{N-1}$ sigma models

Using elementary examples, we will illustrate the concept of constructing surfaces associated with the $\mathbb{C}P^{N-1}$ model.

7.1. Examples of holomorphic solutions of the $\mathbb{C}P^2$ sigma model

From the form of the $\mathbb{C}P^2$ model, it is readily seen that the holomorphic functions are solutions of the $\mathbb{C}P^2$ model. We now concentrate on the following class of holomorphic solutions of the $\mathbb{C}P^2$ model:

$$w_1 = a_1\xi^m, \qquad w_2 = a_2\xi^n, \tag{114}$$

where a_1 and a_2 are complex constants and m and n are real constants. For holomorphic solutions $J = 0$ and the induced metric is conformal. Using the solutions in (114), that metric is given by

$$I = \frac{|a_1|^2|\xi|^{2m}(m^2 + |a_2|^2(m-n)^2|\xi|^{2n}) + |a_2|^2n^2|\xi|^{2n}}{|\xi|^2(1 + |a_1|^2|\xi|^{2m} + |a_2|^2|\xi|^{2n})^2}\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}. \tag{115}$$

The Gaussian curvature $\mathcal{K}$ is computed from (28). After simplification,

$$\mathcal{K} = 4 - \frac{2|a_1|^2|a_2|^2m^2n^2(m-n)^2|\xi|^{2(m+n)}(1 + |a_1|^2|\xi|^{2m} + |a_2|^2|\xi|^{2n})^3}{\left(|a_1|^2|\xi|^{2m}(m^2 + |a_2|^2(m-n)^2|\xi|^{2n}) + |a_2|^2n^2|\xi|^{2n}\right)^3}. \tag{116}$$

In general, $\mathcal{K}$ is not constant. However, $\mathcal{K}$ is constant for certain values of a_1, a_2, m and n. For example, if the second term in (116) vanishes or equals to a constant, then the surfaces associated with the holomorphic solutions (114) of the $\mathbb{C}P^2$ model will have constant Gaussian curvature. This happens when

(i) $a_1 = 0, a_2 = 0, m = 0, n = 0$ and $m = n$ or a combination thereof. For these choices the second term in (116) vanishes; or

(ii) $n = 2m$ and $|a_1|^2 = \pm 2|a_2|$ simultaneously. The second term in (116) then reduces to a constant.

Not surprisingly, constant Gaussian curvature occurs when $a_1 = 0$ or $a_2 = 0$ because the $\mathbb{C}P^2$ model then reduces to the $\mathbb{C}P^1$ model. Hence, the surfaces must have constant Gaussian curvature.

We now consider a case of constant Gaussian curvature surfaces associated with specific holomorphic solutions (114) of the $\mathbb{C}P^2$ model. For simplicity, we take

$$w_1 = \xi, \qquad w_2 = \tfrac{1}{2}\xi^2. \tag{117}$$

The first fundamental form and the Gaussian curvature then are

$$I = \frac{4}{(2 + |\xi|^2)^2}\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}, \qquad \mathcal{K} = 2. \tag{118}$$

Upon substitution of (117) into (112), the coordinates of the radius vector $\vec{X}$ become

$$X_1 = \frac{|\xi|^2(\xi + \bar{\xi})}{(2 + |\xi|^2)^2}, \qquad X_2 = \mathrm{i}\frac{|\xi|^2(\bar{\xi} - \xi)}{(2 + |\xi|^2)^2}, \qquad X_3 = \frac{|\xi|^2(4 - |\xi|^2)}{2(2 + |\xi|^2)^2},$$
$$X_4 = -\frac{\sqrt{3}}{2}\left(1 - \frac{4}{(2 + |\xi|^2)^2}\right), \qquad X_5 = -\mathrm{i}\frac{2(\xi - \bar{\xi})}{(2 + |\xi|^2)^2}, \tag{119}$$
$$X_6 = -\mathrm{i}\frac{(\xi^2 - \bar{\xi}^2)}{(2 + |\xi|^2)^2}, \qquad X_7 = -\frac{2(\xi + \bar{\xi})}{(2 + |\xi|^2)^2}, \qquad X_8 = -\frac{(\xi^2 + \bar{\xi}^2)}{(2 + |\xi|^2)^2}.$$

Of course, the above coordinates satisfy the relation (113). Hence, the surface associated with the specific solutions (117) of the $\mathbb{C}P^2$ model is an affine sphere.

7.2. Mixed solutions of the $\mathbb{C}P^2$ sigma model

In this subsection we analyze the mixed solutions of the $\mathbb{C}P^2$ model and give the first fundamental form, Gaussian curvature and the Weierstrass data for a specific example. It is well known [27] that if the $\mathbb{C}P^2$ model is defined over S^2 and the finiteness of the action (8) is required, then the solutions of the $\mathbb{C}P^2$ model split into three cases: holomorphic solutions, anti-holomorphic solutions and mixed ones. Among these, the mixed solutions can be constructed either from the holomorphic or anti-holomorphic solutions according to the following procedure [6, 27].

Consider three arbitrary holomorphic functions $g_i = g_i(\xi)$, $i = 1, 2, 3$, and define the Wronskian

$$G_{ij} = g_i \partial g_j - g_j \partial g_i, \qquad i = 1, 2, 3 \tag{120}$$

based on any pair. It can be verified that the functions

$$f_i = \sum_{k \neq i}^{3} \bar{g}_k G_{ki}, \qquad i = 1, 2, 3 \tag{121}$$

are solutions of the $\mathbb{C}P^2$ model. The mixed solutions are associated with the ratios

$$w_1 = \frac{f_1}{f_3}, \qquad w_2 = \frac{f_2}{f_3}. \tag{122}$$

Likewise, mixed solutions can be obtained from anti-holomorphic solutions by using $\bar{\partial}$ instead of ∂.

We now continue with the holomorphic functions

$$g_1 = 1, \qquad g_2 = \mathrm{sech}(\xi), \qquad g_3 = \tanh(\xi). \tag{123}$$

Using the above procedure, the mixed solutions of the $\mathbb{C}P^2$ model are

$$w_1 = \tanh\left(\frac{\xi - \bar{\xi}}{2}\right), \qquad w_2 = -\frac{\tanh(\xi) + \tanh(\bar{\xi})}{\mathrm{sech}(\xi) + \mathrm{sech}(\bar{\xi})}, \tag{124}$$

which are of soliton type. These fields satisfy the equations of the $\mathbb{C}P^2$ model. $J = 0$ for this case, as can be readily verified. Hence, the induced metric is conformal and given by

$$I = \frac{2}{1 + \cosh(\xi + \bar{\xi})}\, \mathrm{d}\xi\, \mathrm{d}\bar{\xi}. \tag{125}$$

Note that holomorphicity of the solutions of the $\mathbb{C}P^{N-1}$ model implies that $J = 0$. The converse is false as seen from the above example (124).

The Gaussian curvature is computed from the formula given in (28) (since $J = 0$) and found to be

$$\mathcal{K} = 1. \tag{126}$$

After substituting the solutions (124) into (109) for the $\mathbb{C}P^2$ model, the Weierstrass representation becomes

$$\begin{aligned} &\mathrm{d}X_1 = -\frac{\sinh(\bar{\xi})}{1+\cosh(\xi+\bar{\xi})}\mathrm{d}\xi + \text{c.c.}, \qquad \mathrm{d}X_6 = \mathrm{i}\left[\frac{\cosh(\bar{\xi})}{1+\cosh(\xi+\bar{\xi})}\mathrm{d}\xi - \text{c.c.}\right], \\ &\mathrm{d}X_7 = -\frac{1}{1+\cosh(\xi+\bar{\xi})}\mathrm{d}\xi + \text{c.c.}, \end{aligned} \tag{127}$$

and

$$\mathrm{d}X_2 = 0, \qquad \mathrm{d}X_3 = 0, \qquad \mathrm{d}X_4 = 0, \qquad \mathrm{d}X_5 = 0, \qquad \mathrm{d}X_8 = 0. \tag{128}$$

Integrating (127), we obtain the coordinates of the radius vector $\vec{X}$:

$$\begin{aligned} X_1 &= \operatorname{sech}\left(\frac{\xi+\bar{\xi}}{2}\right)\cosh\left(\frac{\xi-\bar{\xi}}{2}\right), \\ X_6 &= \mathrm{i}\operatorname{sech}\left(\frac{\xi+\bar{\xi}}{2}\right)\sinh\left(\frac{\xi-\bar{\xi}}{2}\right), \\ X_7 &= -\tanh\left(\frac{\xi+\bar{\xi}}{2}\right), \end{aligned} \tag{129}$$

They satisfy $X_1^2 + X_6^2 + X_7^2 = 1$. Hence, the constant Gaussian curvature surface associated with the soliton-like solutions (124) of the $\mathbb{C}P^2$ model is really immersed in $\mathbb{R}^3$ which, in turn, corresponds to the immersion of the $\mathbb{C}P^2$ model into the $\mathbb{C}P^1$ model.

7.3. Examples of surfaces in a low-dimensional $su(N)$ algebra

We briefly discuss the non-splitting solutions $(w_i, \bar{w}_i)$, $i = 1, \ldots, N-1$, of the $\mathbb{C}P^{N-1}$ model invariant under the scaling symmetries $\{S_i\}$ as given in (67). To do so, we subject system (66) to $N-1$ algebraic constraints

$$w_i\bar{w}_i = D_i \in \mathbb{R}, \qquad i = 1, \ldots, N-1. \tag{130}$$

If, for simplicity, we choose $D_i = 1$, then the simplest solutions of this type are

$$w_i = \frac{F_i(\xi)}{\bar{F}_i(\bar{\xi})}, \qquad i = 1, \ldots, N-1, \tag{131}$$

where F_i and $\bar{F}_i$ are arbitrary complex-valued functions of one complex variable each. Substituting (131) into (66), for $N \leqslant 3$ we obtain a class of non-splitting solutions of the $\mathbb{C}P^{N-1}$ model which depend on one arbitrary complex-valued function of one variable ξ and its conjugate. Indeed,

$$w_1 = \frac{F_1(\xi)}{\bar{F}_1(\bar{\xi})}, \qquad w_2 = \frac{c}{\bar{c}}\frac{F_1(\xi)^{\mathrm{e}^{\mathrm{i}\psi}}}{\bar{F}_1(\bar{\xi})^{\mathrm{e}^{-\mathrm{i}\psi}}}, \qquad j = 1, \ldots, N-2, \tag{132}$$

where $c, \bar{c}$ are complex constants and

$$\psi = \pm\frac{\pi}{3} + 2\pi m, \qquad m \in \mathbb{Z}. \tag{133}$$

For brevity, from now on we suppress the subscript 1 and also the arguments of the functions F and $\bar{F}$. For this class of non-splitting solutions, the induced metric g_{ij} has the following components:

$$g_{\xi\xi} = -\frac{N-3}{N^2}\frac{(F')^2}{F^2}, \qquad g_{\bar{\xi}\bar{\xi}} = -\frac{N-3}{N^2}\frac{(\bar{F}')^2}{\bar{F}^2}, \qquad g_{\xi\bar{\xi}} = \frac{2N-3}{N^2}\frac{|F'|^2}{|F|^2}, \tag{134}$$

where prime denotes differentiation with respect to the argument. The determinant of the induced metric then is

$$g = -\frac{3(N-2)}{N^3}\frac{|F'|^4}{|F|^4}. \tag{135}$$

For $N = 2$, the determinant of the induced metric vanishes. Hence, the associated surface for the $\mathbb{C}P^1$ model, subject to the DCs in (130), reduces to a curve in $\mathbb{R}^3$. For $N = 3$, the diagonal components of the induced metric vanish (since $J = 0$). Hence, we have a conformal metric for the $\mathbb{C}P^2$ model subject to the DCs in (130).

From (28) it is straightforward to show that the Gaussian curvature vanishes for the associated surfaces of the $\mathbb{C}P^{N-1}$ model ($N = 3$), subject to the DCs (130). Thus, we conclude that the surfaces associated with solutions of the $\mathbb{C}P^2$ model, which are invariant under dilations, always have zero Gaussian curvature, i.e.,

$$\mathcal{K} = 0. \tag{136}$$

Finally, let us give the coordinates of the radius vector $\vec{X}$ for the non-splitting solutions of the $\mathbb{C}P^2$ model. After substituting the non-splitting solutions (132) of the $\mathbb{C}P^2$ model into the Weierstrass representation (109) and subsequent integration, the coordinates of the radius vector $\vec{X}$ in $\mathbb{R}^8$ are

$$\begin{aligned}
X_1 &= \frac{\mathrm{i}}{6\sqrt{3}|c|^2}|F|^{-2\mathrm{e}^{\mathrm{i}\psi}}\left(\bar{c}^2 F - c^2 \bar{F}|F|^{2\mathrm{i}\sqrt{3}}\right),\\
X_2 &= -\frac{1}{6\sqrt{3}|c|^2}|F|^{-2\mathrm{e}^{\mathrm{i}\psi}}\left(\bar{c}^2 F + c^2 \bar{F}|F|^{2\mathrm{i}\sqrt{3}}\right),\\
X_3 &= \frac{1}{6}\big((1-\mathrm{i}\sqrt{3})\ln F + (1+\mathrm{i}\sqrt{3})\ln \bar{F}\big),\\
X_4 &= -\frac{1}{6}\big((\mathrm{i}+\sqrt{3})\ln F + (-\mathrm{i}+\sqrt{3})\ln \bar{F}\big),\\
X_5 &= -\frac{F^2+\bar{F}^2}{6\sqrt{3}|F|^2},\\
X_6 &= \frac{1}{6\sqrt{3}|c|^2}|F|^{-2\mathrm{e}^{\mathrm{i}\psi}}\left(\bar{c}^2 \bar{F} + c^2 F|F|^{2\mathrm{i}\sqrt{3}}\right),\\
X_7 &= \frac{\mathrm{i}(F^2-\bar{F}^2)}{6\sqrt{3}|F|^2},\\
X_8 &= \frac{\mathrm{i}}{6\sqrt{3}|c|^2}|F|^{-2\mathrm{e}^{\mathrm{i}\psi}}\left(\bar{c}^2 \bar{F} - c^2 F|F|^{2\mathrm{i}\sqrt{3}}\right),
\end{aligned} \tag{137}$$

where ψ is given in (133) and c is a complex constant. The corresponding first fundamental form is immediately obtained from (134) for $N = 3$ and given as

$$I = \frac{2}{3}\frac{|F'|^2}{|F|^2}\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}. \tag{138}$$

8. Summary and concluding remarks

The objective of this paper was to revise and expand on theoretical results in [6] concerning surfaces related to the $\mathbb{C}P^{N-1}$ sigma model. For example, proposition 4 in [6] concerning the structural equations for the $\mathbb{C}P^2$ model (where only the holomorphic solutions were assumed) has been restated as proposition 2. In doing so, we covered in greater detail the geometrical aspects of surfaces immersed in the $su(N)$ algebra. Furthermore, we have derived the formulae in terms of explicit functions in the $\mathbb{C}P^{N-1}$ model, which makes the results in [6] more transparent and useful.

We also computed the Lie-point symmetries of the $\mathbb{C}P^{N-1}$ model equations for arbitrary N. The resulting symmetry algebra is decomposed as a direct sum of two infinite-dimensional simple Lie algebras and the $su(N)$ algebra. Using the Lie-point symmetries, the method of symmetry reduction can now be applied to find solutions which are invariant under subgroups of $SU(N)$ with generic orbits of codimension one. In [38], this analysis was carried out for $N = 2$. The obtained invariant solutions are complicated expressions in terms of elliptic functions. As was shown in [38], for some cases the reduced ordinary differential equations (ODEs) can be transformed into the standard form of the P3 Painlevé transcendent. Matters get worse when $N \geqslant 3$. Although the reduction can still be carried out, the resulting ODEs are coupled and do not appear to be separable. One can prove the existence of solutions but 'how to find them' remains an open problem.

For the $\mathbb{C}P^2$ model, we characterized the immersion of surfaces in the $su(3)$ algebra. Explicit formulae were found for the moving frame, the structural equations (Gauss–Weingarten and Gauss–Codazzi), the first and second fundamental forms, the Gaussian, the mean curvatures, the Willmore functional and the topological charge. These quantities are expressed in terms of holomorphic fields of the $\mathbb{C}P^2$ model. The theoretical concepts have been illustrated with various examples. We also have shown that non-degenerate affine surfaces in $\mathbb{R}^8$ associated with the $\mathbb{C}P^2$ model are affine spheres. Finally, we discussed dilation-invariant solutions of the $\mathbb{C}P^{N-1}$ model, holomorphic immersion of surfaces associated with $\mathbb{C}P^2$ models and mixed soliton-type solutions of the $\mathbb{C}P^2$ model and its corresponding surfaces.

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Appendix

In this appendix we give the explicit form of the vector normals,

$$\eta_j = \phi^\dagger s_j \phi, \qquad j = 3, \ldots, 8,$$

to the surface immersed in the $su(3)$ algebra. The general expressions are too complicated to be useful. Instead, we consider the case of a 2D surface associated with the $\mathbb{C}P^2$ model with solution (117).

We present the normals in the equivalent matrix form.

The first normal is

$$\eta_3 = \phi^\dagger s_3 \phi = \mathrm{i}\eta^3_{ij},$$

where

$$\eta^3_{11} = \frac{4(|\xi|^2-1)}{\Gamma_2^2}, \qquad \eta^3_{12} = \frac{2\xi\left(4+|\xi|^2\Gamma_1\right)}{\Gamma_1\Gamma_2^2}, \qquad \eta^3_{13} = \frac{2\xi^2\Gamma_5}{\Gamma_1\Gamma_2^2},$$
$$\eta^3_{21} = \frac{2\bar{\xi}\left(4+|\xi|^2\Gamma_1\right)}{\Gamma_1\Gamma_2^2}, \qquad \eta^3_{22} = \frac{4+|\xi|^4\left(5+|\xi|^2\Gamma_2\right)}{\Gamma_1^2\Gamma_2^2},$$
$$\eta^3_{23} = -\frac{4\xi(|\xi|^2-1)}{\Gamma_1^2\Gamma_2^2}, \qquad \eta^3_{31} = \frac{2\bar{\xi}^2\Gamma_5}{\Gamma_1\Gamma_2^2}, \tag{A.1}$$
$$\eta^3_{32} = -\frac{4\bar{\xi}(|\xi|^2-1)}{\Gamma_1^2\Gamma_2^2}, \qquad \eta^3_{33} = \frac{|\xi|^2\left(4-|\xi|^2\Gamma_3^2\right)}{\Gamma_1^2\Gamma_2^2},$$

with Γ_j $(j = 1, \ldots, 5)$ defined as

$$\Gamma_j = j + |\xi|^2, \qquad j = 1, \ldots, 5. \tag{A.2}$$

The second normal is

$$\eta_4 = \phi^\dagger s_4 \phi = \mathrm{i}\eta^4_{ij},$$

where

$$\eta^4_{11} = \frac{2\left(2+|\xi|^2(2-|\xi|^2)\right)}{\sqrt{3}\Gamma_2^2}, \qquad \eta^4_{12} = \frac{2\sqrt{3}|\xi|^2\xi}{\Gamma_2^2},$$
$$\eta^4_{13} = -\frac{2\sqrt{3}\xi^2}{\Gamma_2^2}, \qquad \eta^4_{21} = \frac{2\sqrt{3}|\xi|^2\bar{\xi}}{\Gamma_2^2},$$
$$\eta^4_{22} = \frac{4+|\xi|^2(|\xi|^2-8)}{\sqrt{3}\Gamma_2^2}, \qquad \eta^4_{23} = \frac{4\sqrt{3}\xi}{\Gamma_2^2}, \tag{A.3}$$
$$\eta^4_{31} = -\frac{2\sqrt{3}\bar{\xi}^2}{\Gamma_2^2}, \qquad \eta^4_{32} = \frac{4\sqrt{3}\bar{\xi}}{\Gamma_2^2},$$
$$\eta^4_{33} = \frac{|\xi|^2\Gamma_4 - 8}{\sqrt{3}\Gamma_2^2}.$$

The next one is

$$\eta_5 = \phi^\dagger s_5 \phi = \mathrm{i}\mathrm{e}^{-\frac{3\mathrm{i}\varphi}{2}}\eta^5_{ij},$$

where

$$\eta^5_{11} = \frac{2|\xi|(\mathrm{e}^{3\mathrm{i}\varphi}\xi^2-\bar{\xi}^2)}{\Gamma_2^2}, \qquad \eta^5_{12} = -\frac{\sqrt{\xi}\left(4\mathrm{e}^{3\mathrm{i}\varphi}\xi^2\Gamma_1+\bar{\xi}^2(2+|\xi|^2\Gamma_1)\right)}{\sqrt{\bar{\xi}}\Gamma_1\Gamma_2^2},$$
$$\eta^5_{13} = \frac{2\xi^{(3/2)}(2\mathrm{e}^{3\mathrm{i}\varphi}\xi\Gamma_1-\bar{\xi}^3)}{\bar{\xi}^{(3/2)}\Gamma_1\Gamma_2^2}, \qquad \eta^5_{21} = \frac{\sqrt{\bar{\xi}}\left(4\bar{\xi}^2\Gamma_1+\mathrm{e}^{3\mathrm{i}\varphi}\xi^2(2+|\xi|^2\Gamma_1)\right)}{\sqrt{\xi}\Gamma_1\Gamma_2^2},$$
$$\eta^5_{22} = -\frac{2(\mathrm{e}^{3\mathrm{i}\varphi}\xi^2-\bar{\xi}^2)(2+|\xi|^2\Gamma_1)}{|\xi|\Gamma_1\Gamma_2^2}, \qquad \eta^5_{23} = \frac{2\sqrt{\xi}\left(2\bar{\xi}^3+\mathrm{e}^{3\mathrm{i}\varphi}\xi(2+|\xi|^2\Gamma_1)\right)}{\bar{\xi}^{(3/2)}\Gamma_1\Gamma_2^2}, \tag{A.4}$$
$$\eta^5_{31} = \frac{2\bar{\xi}^{(3/2)}\left(\mathrm{e}^{3\mathrm{i}\varphi}\xi^3-2\bar{\xi}\Gamma_1\right)}{\xi^{(3/2)}\Gamma_1\Gamma_2^2}, \qquad \eta^5_{32} = -\frac{2\sqrt{\bar{\xi}}\left(2\mathrm{e}^{3\mathrm{i}\varphi}\xi^3+\bar{\xi}(2+|\xi|^2\Gamma_1)\right)}{\xi^{(3/2)}\Gamma_1\Gamma_2^2},$$
$$\eta^5_{33} = \frac{4(\mathrm{e}^{3\mathrm{i}\varphi}\xi^2-\bar{\xi}^2)}{|\xi|\Gamma_1\Gamma_2^2}.$$

Normal η_6 is given by

$$\eta_6 = \phi^\dagger s_6 \phi = \mathrm{i}\mathrm{e}^{-\frac{3\mathrm{i}\varphi}{2}}\eta^6_{ij},$$

where

$$\begin{aligned}
\eta^6_{11} &= -\frac{2|\xi|\left(e^{3i\varphi}\xi - \bar{\xi}\right)}{\Gamma_2^2}, & \eta^6_{12} &= \frac{2\xi^{(3/2)}\left(2e^{3i\varphi}\Gamma_1 - \bar{\xi}^2\right)}{\sqrt{\bar{\xi}}\Gamma_1\Gamma_2^2},\\
\eta^6_{13} &= -\frac{\xi^{(3/2)}\left(4e^{3i\varphi}\Gamma_1 + |\xi|^2\bar{\xi}^2\Gamma_3\right)}{\bar{\xi}^{(3/2)}\Gamma_1\Gamma_2^2}, & \eta^6_{21} &= -\frac{2\bar{\xi}^{(3/2)}\left(2 - e^{3i\varphi}\xi^2 + 2|\xi|^2\right)}{\sqrt{\xi}\Gamma_1\Gamma_2^2},\\
\eta^6_{22} &= -\frac{4|\xi|\left(e^{3i\varphi}\xi - \bar{\xi}\right)}{\Gamma_1\Gamma_2^2}, & \eta^6_{23} &= \frac{2\xi^{(3/2)}\left(2e^{3i\varphi} + \bar{\xi}^2\Gamma_3\right)}{\sqrt{\bar{\xi}}\Gamma_1\Gamma_2^2},\\
\eta^6_{31} &= \frac{\bar{\xi}^{(3/2)}\left(4 + 4|\xi|^2 + e^{3i\varphi}|\xi|^2\xi^2\Gamma_3\right)}{\xi^{(3/2)}\Gamma_1\Gamma_2^2}, & \eta^6_{32} &= -\frac{2\bar{\xi}^{(3/2)}\left(2 + e^{3i\varphi}\xi^2\Gamma_3\right)}{\sqrt{\xi}\Gamma_1\Gamma_2^2},\\
\eta^6_{33} &= \frac{2|\xi|\left(e^{3i\varphi}\xi - \bar{\xi}\right)\Gamma_3}{\Gamma_1\Gamma_2^2}.
\end{aligned} \tag{A.5}$$

Normal η_7 is given by

$$\eta_7 = \phi^\dagger s_7 \phi = e^{-\frac{3i\varphi}{2}}\eta^7_{ij},$$

where

$$\begin{aligned}
\eta^7_{11} &= -\frac{2|\xi|\left(e^{3i\varphi}\xi^2 + \bar{\xi}^2\right)}{\Gamma_2^2}, & \eta^7_{12} &= \frac{\sqrt{\xi}\left(4e^{3i\varphi}\xi^2\Gamma_1 - \bar{\xi}^2(2 + |\xi|^2\Gamma_1)\right)}{\sqrt{\bar{\xi}}\Gamma_1\Gamma_2^2},\\
\eta^7_{13} &= -\frac{2\xi^{(3/2)}\left(\bar{\xi}^3 + 2e^{3i\varphi}\xi\Gamma_1\right)}{\bar{\xi}^{(3/2)}\Gamma_1\Gamma_2^2}, & \eta^7_{21} &= \frac{\sqrt{\bar{\xi}}\left(4\bar{\xi}^2\Gamma_1 - e^{3i\varphi}\xi^2(2 + |\xi|^2\Gamma_1)\right)}{\sqrt{\xi}\Gamma_1\Gamma_2^2},\\
\eta^7_{22} &= \frac{2\left(e^{3i\varphi}\xi^2 + \bar{\xi}^2\right)\left(2 + |\xi|^2\Gamma_1\right)}{|\xi|\Gamma_1\Gamma_2^2}, & \eta^7_{23} &= \frac{2\sqrt{\xi}\left(2\bar{\xi}^3 - e^{3i\varphi}\xi(2 + |\xi|^2\Gamma_1)\right)}{\bar{\xi}^{(3/2)}\Gamma_1\Gamma_2^2},\\
\eta^7_{31} &= -\frac{2\bar{\xi}^{(3/2)}\left(e^{3i\varphi}\xi^3 + 2\bar{\xi}\Gamma_1\right)}{\xi^{(3/2)}\Gamma_1\Gamma_2^2}, & \eta^7_{32} &= \frac{2\sqrt{\bar{\xi}}\left(2e^{3i\varphi}\xi^3 - \bar{\xi}(2 + |\xi|^2\Gamma_1)\right)}{\xi^{(3/2)}\Gamma_1\Gamma_2^2},\\
\eta^7_{33} &= -\frac{4\left(e^{3i\varphi}\xi^2 + \bar{\xi}^2\right)}{|\xi|\Gamma_1\Gamma_2^2}.
\end{aligned} \tag{A.6}$$

The last normal is given by

$$\eta_8 = \phi^\dagger s_8 \phi = e^{-\frac{3i\varphi}{2}}\eta^8_{ij},$$

where

$$\begin{aligned}
\eta^8_{11} &= \frac{2|\xi|\left(e^{3i\varphi}\xi + \bar{\xi}\right)}{\Gamma_2^2}, & \eta^8_{12} &= -\frac{2\xi^{(3/2)}\left(\bar{\xi}^2 + 2e^{3i\varphi}\Gamma_1\right)}{\sqrt{\bar{\xi}}\Gamma_1\Gamma_2^2},\\
\eta^8_{13} &= \frac{\xi^{(3/2)}\left(4e^{3i\varphi}\Gamma_1 - |\xi|^2\bar{\xi}^2\Gamma_3\right)}{\bar{\xi}^{(3/2)}\Gamma_1\Gamma_2^2}, & \eta^8_{21} &= -\frac{2\bar{\xi}^{(3/2)}\left(2 + e^{3i\varphi}\xi^2 + 2|\xi|^2\right)}{\sqrt{\xi}\Gamma_1\Gamma_2^2},\\
\eta^8_{22} &= \frac{4|\xi|\left(e^{3i\varphi}\xi + \bar{\xi}\right)}{\Gamma_1\Gamma_2^2}, & \eta^8_{23} &= -\frac{2\xi^{(3/2)}\left(2e^{3i\varphi} - \bar{\xi}^2\Gamma_3\right)}{\sqrt{\bar{\xi}}\Gamma_1\Gamma_2^2},\\
\eta^8_{31} &= \frac{\bar{\xi}^{(3/2)}\left(4 + 4|\xi|^2 - e^{3i\varphi}|\xi|^2\xi^2\Gamma_3\right)}{\xi^{(3/2)}\Gamma_1\Gamma_2^2}, & \eta^8_{32} &= \frac{2\bar{\xi}^{(3/2)}\left(e^{3i\varphi}\xi^2\Gamma_3 - 2\right)}{\sqrt{\xi}\Gamma_1\Gamma_2^2},\\
\eta^8_{33} &= -\frac{2|\xi|\left(e^{3i\varphi}\xi + \bar{\xi}\right)\Gamma_3}{\Gamma_1\Gamma_2^2}.
\end{aligned} \tag{A.7}$$

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